

A variable kinematic one-dimensional model for aeroelasticity and dynamic analysis of multi-layered rotors

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POLITECNICO DI TORINO

SCUOLA DI DOTTORATO

Dottorato di Ricerca in Fluidodinamica - XXVII ciclo

PhD Dissertation

**A variable kinematic one-dimensional model for  
aeroelasticity and dynamic analysis of multi-layered  
rotors**



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March 2015



*Alla mia Famiglia.  
Ad Alessia.*



# Ringraziamenti

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# Abstract

Flutter is one of the most known instability phenomena. This condition occurs when a given structure exhibits sustained, harmonic oscillations, sometimes leading to catastrophic events. The prediction of flutter represents a crucial point for a correct and safe design. When fluid-structure interactions produce dynamic instability, flutter analyses require accurate descriptions of body deformations and aerodynamic loads. To this end, aerodynamic theories have been coupled with structural models to develop aeroelastic analysis tools, whose reliability is the results of a trade-off between the accuracy and the computational efficiency.

From a computational point of view, the most efficient formulation is based on the 1D assumption, where the problem is reduced to a set of variables that only depends on the beam-axis coordinate. Besides the well-known classical beam theories, several refined kinematic models have been proposed, to study the stability of rotating blades and shafts. However, when these structures are highly deformable or the material distribution involves non-classical structural couplings, 2D and 3D solutions are still required.

Within this work we propose an advanced 1D formulation to analyse the stability of rotating structures. The higher-order beam theories are obtained using the Carrera Unified Formulation (CUF), which allows to derive, at least theoretically, an infinite number of kinematic models. The Equations of Motion (EoM) for shafts and blades include the Coriolis term and the centrifugal effects (spin softening and geometrical stiffening). For the subsonic flow regime, aerodynamic loads are defined following the unsteady strip theories proposed by Theodorsen and Loewy. For the supersonic regime, the linear Piston theory is extended to structures rotating in compressed air flow. The Finite Element Method (FEM) is used to solve the weak form of the EoM.

Firstly, to evaluate the accuracy of 1D CUF elements, static and free-vibration analysis are carried out on compact and thin-walled structures of isotropic, orthotropic and functionally graded materials. Then, higher-order elements are used to study the dynamics of laminated shafts, thin cylinders, discs and blades, which rotate about the longitudinal and transverse axis. Results show the improved performance of the 1D CUF theories compared to 2D and 3D solutions.

In order to evaluate the proposed aeroelastic formulation, we test different wing configurations, including thin-walled box beams. The effects of the sweep angle and the lamination scheme on flutter conditions are evaluated, and results are compared with plate solutions, experimental tests and aeroelastic analysis carried out with the Doublet Lattice Method (DLM). Moreover, comparisons between Theodorsen and Loewy aerodynamic theories are presented



for a realistic rotary-wing model. In the last numerical examples, the linear Piston Theory is used to describe the dynamics of thin plates with different aspect-ratio surrounded by compressed air. For this cases, results are compared with an existing solution based on a non-linear plate theory.

# Sommario

*Il flutter è sicuramente il più conosciuto fenomeno di instabilità dinamica. Quando tale instabilità si verifica, la struttura presenta armoniche e auto-sostenute vibrazioni che, in casi sfortunati, possono provocare il collasso della stessa. La previsione della condizione di flutter è pertanto un punto cruciale per una corretta e sicura progettazione. Quando l'instabilità è dovuta ad interazioni tra la struttura e il flusso circostante, le analisi aerelastiche richiedono descrizioni accurate del campo di deformazione e delle distribuzioni di forze aerodinamiche a cui il corpo è assoggettato. A tal fine, teorie aerodinamiche e strutturali vengono accoppiate in modelli aeroelastici, le cui affidabilità sono il risultato di un compromesso tra l'accuratezza dei modelli e il loro effettivo costo computazionale.*

*L'approssimazione unidimensionale, certamente la più efficiente, prevede la riduzione del problema reale (definito in tre dimensioni) ad un set di variabili dipendenti da una solo parametro. La dinamica e la stabilità di alberi e superfici portanti in rotazione sono state largamente studiate con l'uso delle teorie trave, concepite secondo le classiche assunzioni di Eulero-Bernoulli e Timoshenko o basate su ipotesi cinematiche meno stringenti. Tuttavia, quando tali strutture sono altamente deformabili o la distribuzione del materiale causa accoppiamenti strutturali tra differenti modi di deformazione, ci si deve affidare alle soluzioni bi- e tridimensionali, con un conseguente aumento del costo computazionale.*

*Questa tesi si propone di fornire una formulazione 1D avanzata per lo studio della stabilità delle strutture rotanti in presenza o meno di un fluido circostante. Le teorie unidimensionali vengono concepite secondo la Carrera Unified Formulation (CUF), la quale consente di derivare un numero teoricamente infinito di modelli cinematici. Le equazioni del moto di alberi e travi rotanti includono, oltre al termine di Coriolis, lo 'spin softening' e la rigidità geometrica dovuta all'effetto centrifugo. I carichi aerodinamici in regime di flusso subsonico vengono definiti tramite le teorie instazionarie proposte da Theodorsen e Loewy. In regime supersonico, invece, le forze aerodinamiche agenti su piastre rotanti in un flusso di fluido compresso vengono calcolate secondo la 'piston theory' lineare. Le equazioni di governo vengono risolte con il metodo degli elementi finiti.*

*Al fine di valutare l'accuratezza degli elementi CUF 1D, numerose analisi statiche e di vibrazioni libere sono state effettuate su strutture in materiale isotropo, ortotropo e funzionalmente graduato. Successivamente, gli stessi elementi sono utilizzati per valutare la dinamica di una vasta gamma di strutture rotanti, comprendente alberi compatti laminati, cilindri e dischi sottili e pale. I risultati sono confrontati con soluzioni bi- e tridimensionali, confermando l'efficacia dell'approccio adottato.*

*Il modello aeroelastico proposto viene testato considerando diverse configurazioni alari, tra cui travi laminate in materiale ortotropo a parete sottile. Gli effetti dell'angolo di freccia e dello schema di laminazione sulle condizioni di flutter vengono attentamente analizzati. I risultati proposti sono confrontati con soluzioni bidimensionali, evidenze sperimentali e con valori ottenuti tramite l'impiego del Dou-*

*blet Lattice Method. Inoltre, viene studiata la risposta aeroelastica di una realistica pala di elicottero in rotazione tramite gli approcci instazionari di Theodorsen (esteso al caso rotante) e Loewy. Infine, è presentato lo studio aeroelastico di piastre rotanti in un flusso supersonico di aria compressa.*

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# Chapter 1

## Introduction

*According to Collar's definition, aeroelasticity is "the study of the mutual interaction that takes place within the triangle of the inertial, elastic, and aerodynamic forces acting on structural members exposed to an airstream, and the influence of this study on design". In the past, the fluid-structure interaction (FSI) has brought to catastrophic events due to sudden failures of bridges, airplanes, helicopters etc. A correct and safe design must require, therefore, an accurate prediction of aeroelastic phenomena. Unfortunately, FSI analyses are, in most cases, too computationally expensive, hence a trade off between accuracy and cost is often mandatory.*

*This thesis aims to present simple tools for the study of the dynamics and stability of shafts and blades. Simple aeroelastic formulations based on higher-order beam theories and unsteady strip approaches are proposed for flutter predictions.*

### 1.1 Flutter and aeroelastic tools

Flutter is defined as the dynamic instability of an elastic body in an airstream that is subjected to large lateral aerodynamic loads and it is one of the most important aeroelastic phenomena. The flutter condition, defined by the critical speed ( $V_F$ ) and by a circular frequency ( $\omega_F$ ) occurs when a given structure exhibits sustained, harmonic oscillations. In order to predict the borderline condition, it is paramount to know the interactions between elastic deformations and aerodynamic loads imposed by fluid motion. To this end, many aeroelastic theories have been developed. The aeroelastician usually needs information on two- and three-dimensional external flows and, in order to simplify the mathematical theories, a number of assumptions are usually made.

#### Fixed-wing aeroelasticity

A number of aerodynamic approaches was presented in the first half of the twentieth century by important aerodynamicists such as Cicala (Cicala, 1934-1935), Ellenberger (Ellenberger, 1936), Kussner (Kussner, 1936), Burgers with von Karman (von Karman and Burgers, 1943) and, in particular, Theodorsen (Theodorsen, 1935). The proposed strip theories and their modifications, properly combined with simplified structural models, have allowed simple but reliable tools to be developed. For instance, on the basis of beam-plate approach, in which the bending stiffness, the torsional stiffness and the secondary stiffnesses (bending-torsion, warping etc.) are reduced to the beam equivalent stiffnesses ( $EI$ ,  $GJ$ ,  $K$  etc.), wings (Guo et al., 2003, 2006; Qin

and Librescu, 1985) and bridge decks (Banerjee, 2003) were accurately studied. Unfortunately, this structural approach is no longer considered reliable when further couplings occur due to the distribution of the materials. For this reason, more refined theories have been developed. Over the years, Librescu *et al.* considered various aspects related to FSI such as divergence conditions of swept-forward wings (Librescu and Khdeir, 1988), effects of external stores (Gem and Librescu, 1998) and aeroelastic tailoring of composite structures (Gem and Librescu, 2000)-(Qin and Librescu, 2003) on both static and dynamic stability, the response to gust (Qin *et al.*, 2002) and blast loadings (Marzocca *et al.*, 2002). In most of these papers, the wing cross-section was either rectangular or biconvex, and the non-classical effects, such as warping deformations, transverse shear and non-uniform torsion were included in a sophisticated beam theory. Recently, the same structural model was used in (Ji-Seok *et al.*, 2013) and (Masjedi and Ovesy, 2014) where an attempt was made to suppress the instability and a number of interesting considerations on the different structural constitutive assumptions was presented, respectively. In particular, since the computation of the bending and torsional stiffnesses depends on the choice of the constitutive equations, Masjedi and Ovesy (2014) showed that the results of aeroelastic analyses can differ to a great extent, even as much as 40%. Moreover, other improvements were proposed by Patil *et al.* in (Cesnik *et al.*, 1996)-(Patil, 1997). The cross-sectional stiffnesses of composite box beams were first computed by means of an asymptotically correct formulation and then included in a non-linear beam modeling framework. The effectiveness of this approach was pointed out in (Palacios and Epureanu, 2011), where Palacios *et al.* dealt with the aeroelastic behaviour of highly flexible wings.

In the 1970s, the development of three dimensional aerodynamic panel/lifting surface methods enabled prediction of flutter conditions and gust responses of new structural configurations. For instance, the Doublet Lattice Method (DLM)(Albano and Rodden, 1969) and its refinements (Rodden *et al.*, 1998) were extensively used for the study of non-planar configurations (Kalman *et al.*, 1970), non-conventional planform shaped wings (Lan, 1979) and complex wing/tail interactions in both subsonic and supersonic flow (Yurkovich, 2003). Panel theories have been usually combined with bi-dimensional structural models for the description of the above complex structures. Nowadays, the rapid increase of the computational power has motivated the development in the area of computational fluid dynamics (CFD). Over the last 30 years, the CFD-based aeroelasticity progressed from full potential theories to problems governed by the Navier-Stokes equations. Recent works offer interesting overviews of CFD-aeroelastic tools (Bennet and Edwards, 1998; Huttshell *et al.*, 2001) furthermore providing useful information about the emerging trends in the aeroelasticity field (Livne, 2003). From a structural point of view, the CFD simulations can be coupled with finite element models, in which non-linear elements may be used in order to include either geometrical effects or large deformations. A detailed paper about non-linear structural models can be found in Demasi and Livne (2009).

## **Rotating blade aeroelasticity**

As far as the aeroelasticity of rotary-wings is concerned, the basic problem becomes more complex due to the geometric nonlinearities that must be taken into account in both elastic and aerodynamic terms. This need has driven the development of suitable structural and aerodynamic theories as confirmed by the considerable number of articles available in the literature. Almost 50 years ago, Loewy (1969) provided a thorough overview on a wide range of topics related to the dynamics and aeroelasticity of rotary-wings such as flap-lag flutter, pitch-lag flut-

ter and ground and air resonance. Although being more limited in scope, other contributions on the study of classical flutter conditions and unsteady aerodynamic theories were proposed by Ham (1973) and Dat (1973), respectively. In that period, the aerodynamic models for the instability predictions were basically extensions of steady and quasi-steady strip theories conceived for fixed-wings. The first important rotary-wing unsteady approach for hover, based on Theodorsen's model, was proposed by Loewy (1957). Although in an approximate manner, this theory takes into account the effect of the spiral returning wake beneath the rotor. More details about Loewy's theory will be provided in next sections. Other valuable aerodynamic models were proposed on the basis of Greenberg's theory (Greenberg, 1947). According to this approach, a pulsating velocity variation and a constant pitch angle were included in Theodorsen's model. These modifications were extended to the case of rotary-wings by Friedmann and Yuan (1977), in which Theodorsen's theory, Loewy's theory for both incompressible and compressible flows and Possio's theory were used for the coupled flap-lag-torsional aeroelastic problem of a rotor blade in hover. The authors pointed out that the above modifications should be included to realistically reproduce the aeroelasticity of rotary-wings, where the assumptions commonly used in deriving strip theories (1 - cross sections are assumed to perform only simple harmonic pitching and plunging oscillations about a zero equilibrium position; 2 - the velocity of oncoming airflow is constant; 3 - usual potential small-disturbance unsteady aerodynamics are assumed to apply) are intrinsically violated. The modified strip theory with the quasi-steady approximation was used to study the stability of composite helicopter blades with swept tips both in hover and in forward flight (Yuan and Friedmann, 1995). The swept-tip blades were modeled by means of non-linear 1D finite element with the inclusion of transverse shear deformations and out-of-plane warping. These results were recently used to verify the accuracy of a nonlinear structural formulation valid for slender, homogeneous and twisted blades (Piccione et al., 2010). Other analyses on advanced geometry blades were carried out in (Bir and Chopra, 1994) with the purpose of evaluating the effects of sweep and droop on both rotor aeroelastic stability and rotorcraft aeromechanical stability. The evolution of the mechanics of helicopter blades, focusing on the aeroelastic and aerodynamic issues, was extensively discussed in (Conlisk, 1997; Friedmann, 2004; Friedmann and Hodges, 1977, 2003). Although the aeroelasticity of large wind turbine blades is inherently different, several related aeroelastic problems were solved using rotary-wing theories (Friedmann, 1976; Kaza and C.E., 1976, May; Ormiston, 1973). Also in this case, interesting reviews are available (Friedmann, 1980; Hansen et al., 2006; Rasmussen et al., 2003), in which the response problems of an isolated wind turbine blade and a complete rotor/tower configuration were detailed discussed.

## 1.2 Structural models for rotary-wings

A thorough understanding of the dynamics of rotating blades is the starting point for the study of fatigue effects, forced-response and flutter instability, which occur in airplane engines, helicopters and turbomachinery. The modelling of rotating structures is usually made with beam formulations. Many researchers addressed the problem of the rotating beam by simplifying both equations of motion and displacement formulations. For instance, Banerjee (2000); Banerjee et al. (2006), Ozge and Kaya (2006), Mei (2008) and Hodges and Rutkowski (1981) limited their studies to the flexural vibrations of both uniform and tapered Euler-Bernoulli rotating beams using, respectively, the Dynamic Stiffness Method, the Differential Transform Method and a variable-order finite element approach. The assumption that the beam deforms only in bending mode is restrictive, since the coupling between the axial deformation and the lagwise



motion can be significant. To take into account this coupling the introduction of the Coriolis force becomes mandatory. For this reason, Huang et al. (2010) solved the complete motion equations of Euler beams using the power series solution, and the Finite Element Method (Tsai et al., 2011). Improvements were achieved with the introduction of enhanced displacement fields over the blade cross-section. Several studies were conducted on the development of theories for the rotating structures based on the Timoshenko model, where the Coriolis term was (Stoykov and Ribeiro, 2013) or was not considered (Banerjee, 2001; Lee and Kuo, 1993; Rao and Gupta, 2001; Yoo et al., 2005). In all these works, the generic rotating blade is assumed to be a compact structure constituted by isotropic material or by orthotropic laminae. However, the design of advanced rotor blades was strongly affected by the advent of composite materials, which combine a high specific strength and stiffness with simplified formability. These properties produce light and efficient blades, whose dynamic characteristics usually involve phenomena that cannot be detected by the use of classical models. For this reason, a considerable number of refined theories were introduced with the purpose of describing the rotating composite blade behaviour. For instance, Song *et al.* presented in (Song and Librescu, 1997b) a structural model encompassing transverse shear, secondary warping deriving from the assumption of the non-uniform torsion along the longitudinal axis and the effect of the heterogeneity of the materials. They observed that, by discarding the Coriolis term, the equations described separately the flap-lag deformation and the extension-twist motion, and within this context, they examined the ply orientation effects. Contrary to this ad-hoc formulation, Chandiramani et al. (2002) provided a geometrically nonlinear theory for analyzing the rotating composite thin-walled box beam, in which the non-classical effects were captured in a general way. The linearized equations of motion were solved with the Modified Galerkin Method and the Coriolis term was disregarded. Additionally, the authors extended their formulation to pre-twisted composite blades (Chandiramani et al., 2003). Interesting studies on controlling thin-walled composite blades via piezoelectric patches were presented in (Na et al., 2003; Song et al., 2002). Jung et al. (2001) developed a one-dimensional finite element based on a mixed variational approach in which both displacement and force formulations were used. The walls of the structures considered were modelled as shell and the global deformation was described by the Timoshenko beam model. This model is suitable for composite structures with open and closed contour. Detailed reviews about the dynamics of rotating composite blades are provided in (Hodges, 1990; Jung et al., 1999).

### 1.3 Structural models for spinning shafts

This section aims to provide an overview (even if not comprehensive) of structural theories used to analyse the dynamics of spinning structures.

The classical beam theory proposed by Euler and Bernoulli has been adopted by many researchers. For instance, Bauer (1980) presented an analytical approach to investigate the vibrational behaviour of a beam rotating with constant speed about its longitudinal axis for all boundary conditions. About ten years later, Chen and Liao (1991) used the assumed-modes approximate method to evaluate how the dynamic behavior of a rotating slender beam is affected by the ratio of moments of inertia of the cross-section and the pre-twisted angle. Furthermore, by using the same structural model, Banerjee and Su (2004) solved the differential equations of motion by extending the field of application of the Dynamic Stiffness Method (DSM) to rotating structures. Unfortunately, these solutions are inadequate for short and stubby bodies in which rotary inertia and shear deformations are important. In order to take into account these effects,

more refined theories have been widely employed. In fact, Zu and Han presented analytical solutions for free flexural vibrations of a spinning, finite Timoshenko beam subjected to a moving load for the six classical boundary conditions (Jean Wu-Zheng and P.S. Han, 1992). To consider elastic supports, Choi et al. (2000) presented an efficient and accurate quadrature method to investigate the dynamics of a spinning shearable beam. On the other hand, in the framework of DSM, Curti (Curti et al., 1991), (Curti et al., 1992) analyzed the vibrational behaviour of moderate thick shafts by adding rotatory inertia and shear deformation effects to Euler-Bernoulli's model. In (Sheu and Yang, 2005), the authors showed that both whirl and critical speeds can be analytically expressed as a function of a geometric parameter (slenderness ratio) of a Rayleigh beam. Furthermore, this parameter limits the number of critical speeds.

Several experimental tests demonstrated the advantages of orthotropic materials over their isotropic counterparts (Bauchau, 1981; Zinberg and Symmonds, 1970; Zorzi and Giordano, 1985). As a consequence, the design of rotors has demanded the development of adequate and reliable structural models. For instance, Bert and Kim developed one-dimensional models, based on Euler-Bernoulli (Bert and Kim, 1992) and Bresse-Timoshenko (Bert and Kim, 1995) theories, which included bending-twisting coupling. The critical speeds were in good agreement with the results obtained through both shell and experimental approaches. Naturally, when shear effects become important, results obtained with the Euler-Bernoulli theory are not accurate. In (Chen and Peng, 1998), the authors investigated the stability of composite spinning cylinders subjected to compressive loads using Timoshenko's model and the equivalent modulus beam theory (EMBT). Critical speeds and effects due to a disk location were also studied under these assumptions. Another contribution was made by Chang et al. (2004), who developed a simple first-order shear deformation theory for shafts supported by bearings modeled as springs and viscous dampers. The constitutive relations were derived directly from the equations of 3D continuum mechanics using coordinate transformations.

To overcome the limitations of the equivalent modulus approach, Singh and Gupta (1996) first proposed a refined beam formulation derived from a layerwise shell theory (LBT) and then, in (Gubran and Gupta, 2005), presented a modified EMBT that included bending-twisting, shear-normal and bending-stretching coupling effects. The obtained results were used by Sino et al. (2008) to assess their simplified homogenized beam theory (SHBT), where the stiffness parameters are evaluated using an energy formulation that takes into account several contributions, such as Young's modulus, the shear modulus, the distance from the shaft axis and the thickness of each rotor layer. The results were in good agreement with those presented by Gupta *et al.*. Other attempts have also been made to improve displacement theories. Among these formulations, Librescu *et al.* provided a higher order model which incorporates the warping and torsion of thin-walled anisotropic spinning structures. Critical speeds as well as stability were investigated considering thin-walled boxes (Song and Librescu, 1997a; Song et al., 2000) and cylinders (Na et al., 2006). When the assumptions of beam theories are too restrictive, two- and three-dimensional formulations become useful. For instance, in (Kim and Bert, 1993), the authors compared a number of shell theories (Loo's, Morley's, Love's, Donnell's and Sanders's theories) to determine the critical speeds of the shaft examined in (Zinberg and Symmonds, 1970). All these models yielded accurate results, except Donnell's theory, which was not effective for long shafts. Another example is (Ramezani and Ahmadian, 2009), in which the layerwise theory was combined with the Wave Propagation Approach for rotating laminated shells under different boundary conditions. Interesting results were presented in (Zhao et al., 2002), in which the dynamics of rotating cross-ply laminated cylinders was considered. The structure was reinforced with stringers and rings whose stiffness was considered either

separately or smeared over the shell surface. As expected, the number of stiffeners had a remarkable effect on the backward and forward frequencies. Moreover, in (Chatelet et al., 2002), a disk-shaft assembly was discretized with shell finite elements, considering a cyclic symmetry of the structure; three analyses were carried out with different numbers of degrees of freedom. This assumption allowed a considerable reduction of the computational cost. Comprehensive overviews of rotor dynamic phenomena and classical analysis approaches can be found in (Genta, 2005; Hua et al., 2005; Rao, 1985).

## 1.4 Carrera Unified Formulation

Carrera's Unified Formulation (CUF) offers a procedure to obtain refined structural models, in which the order and the type of theories are free input parameters. CUF was first developed for plate and shell theories. A comprehensive description of possible approaches for laminated plates and shells based on both displacement and transverse stress assumptions were provided in Carrera (2002, 2003). Later, Ballhause et al. (2004) investigated the static and dynamic behavior of multilayered plates embedding piezo-layers. The results show that CUF is able to lead to a quasi-3D description of the global and local characteristics of piezoelectric plates. Other assessments were presented in (Carrera, 2004), where the vibrational free response of homogeneous and multilayered simply supported plates was evaluated. Carrera et al. (2007) used CUF to analyze multi-field problems related to multilayered plates subjected to thermal, mechanical and electric loads. In (Carrera et al., 2008), same analyses were performed also considering a magnetic load. Moreover, Carrera and Giunta carried out failure analyses for simply supported, isotropic and orthotropic plates using the failure criteria developed by Von Mises (Carrera and Giunta, 2008), Tsai-Hu, Tsai-Hill, Hoffman and Hashin (Carrera and Giunta, 2009). Regarding functionally graded materials (FGM), Brischetto investigated plates subjected to a transverse mechanical load (Brischetto and Carrera, 2010) using CUF and Reissner's mixed variational theory. Within the same topic, Neves *et al.* provided static, free vibration and buckling analysis of FGM plates (Neves et al., 2012b) and shells (Neves et al., 2012a) by means of a meshless technique.

In the latest years, CUF has been extended to the beam model. Carrera and Giunta tested the performance of higher-order 1D theories, using N-order Taylor type expansions of the section coordinates in order to define the displacement variables (Carrera and Giunta, 2010). The study was carried out on isotropic beams with rectangular and I-shaped cross-sections and the governing differential equations were solved via Navier type closed-form solution. In (Carrera et al., Sept. 2009), displacement unknowns were expanded using N-order Lagrange polynomials, which were defined on a set of sampling points belonging to the section. Carrera et al. (2010a) analysed the static behavior of a beam with an airfoil-like section subjected to a bending load using Taylor-like elements. The authors demonstrated that the use of higher-order theories improves the flexibility of the beam element as well as the accuracy of the displacement, stress and strain distributions over the cross-section. Beam formulation was also employed to analyse the static behavior of thin-walled structures, bridge-like cross-sections (Carrera et al., 2012b), furthermore, to present buckling and static analyses of laminated composite beams (Ibrahim et al., 2012),(Catapano et al., 2011). As for the Taylor-like model, it has been shown that the number of terms that should be retained for each of the considered beam theories is closely related to the problem addressed. A comprehensive discussion of this issue is proposed by Carrera and Petrolo (2011a). As far as free-vibrational analyses are concerned, the dynamic characteristics of different beam sections and wing model are presented in (Carrera et al.,

2011b, 2012a; Petrolo et al., 2012). Other investigations have been carried out on hollow cylindrical reinforced structures (Carrera et al., 2013f), laminated composite (Giunta et al., 2013) and functionally graded beams (Giunta et al., 2011). Besides Taylor- and Lagrange-based expansions, several displacement theories obtained by means of combinations of different functions (trigonometric, exponential, hyperbolic etc.) have been recently tested on laminated structures made of either composite (Carrera et al., 2013b,d) or functionally graded (Daoud et al., 2014) materials. Other encouraging results have also been obtained in the rotordynamics field by analyzing both rotating blades (Carrera et al., 2013c) and spinning shafts (Carrera and Filippi, 2014; Carrera et al., 2013a) (some results are reported in this thesis). Regarding structure-fluid interactions, CUF elements have already been used with both Vortex (Carrera et al., 2013e; Varello et al., 2011, 2013) and Doublet (Pagani et al., 2014; Petrolo, 2012, 2013) Lattice method and, for supersonic flows, with the Piston theory (Carrera and Zappino, 2014b; Carrera et al., 2014c).

Many other papers exploit the features of CUF, but for a thorough and clear description we refer to (Carrera et al., 2014a, 2011a).



## Chapter 2

# Structural models

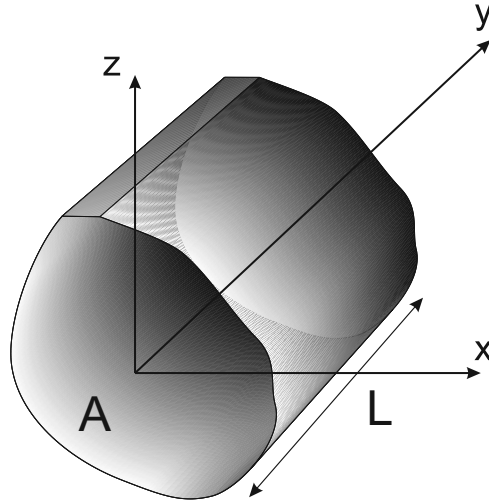
*In this chapter, Carrera's Unified Formulation (CUF) is presented. This formulation allows, at least theoretically, the definition of an infinite number of structural theories going beyond the assumptions of the classical refined beam models. The number and the kind of functions used within the displacement field are input parameters of the formulation and determine the class of the 1D theory. A number of Equivalent-Single Layer (ESL) theories and the Layer-Wise (LW) approach are briefly discussed in the following.*

### 2.1 Preliminaries: geometry and material arrays

We introduce the transposed displacement vector

$$\mathbf{u}(x, y, z) = \{ u_x \quad u_y \quad u_z \}^T \quad (2.1)$$

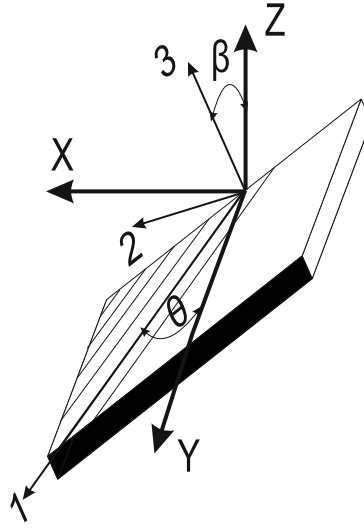
where  $x$ ,  $y$ , and  $z$  are orthonormal axes, as shown in Fig.2.1. The cross-section of the structure



**Figure 2.1** Structural reference system

is  $A$  and the longitudinal axis is  $0 \leq y \leq L$ . The stress,  $\sigma$ , and strain,  $\epsilon$ , components are grouped as

$$\begin{aligned} \sigma_p &= \{ \sigma_{zz} \quad \sigma_{xx} \quad \sigma_{zx} \}^T, & \epsilon_p &= \{ \epsilon_{zz} \quad \epsilon_{xx} \quad \epsilon_{zx} \}^T \\ \sigma_n &= \{ \sigma_{zy} \quad \sigma_{xy} \quad \sigma_{yy} \}^T, & \epsilon_n &= \{ \epsilon_{zy} \quad \epsilon_{xy} \quad \epsilon_{yy} \}^T \end{aligned} \quad (2.2)$$



**Figure 2.2** Physical and material reference system

The subscript "n" stands for those terms lying on the cross-section, while "p" stands for the terms lying on planes which are orthogonal to  $A$ .

The strains are obtained as

$$\begin{aligned}\epsilon_p &= D_p \mathbf{u} \\ \epsilon_n &= D_n \mathbf{u} = (D_{np} + D_{ny}) \mathbf{u}\end{aligned}\quad (2.3)$$

where  $D_p$  and  $D_n$  are differential operators whose explicit expressions are

$$D_p = \begin{bmatrix} 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & 0 & 0 \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{bmatrix}, \quad D_{np} = \begin{bmatrix} 0 & \frac{\partial}{\partial z} & 0 \\ 0 & \frac{\partial}{\partial x} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad D_{ny} = \begin{bmatrix} 0 & 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \end{bmatrix}\quad (2.4)$$

Constitutive laws are introduced to obtain the stress components,

$$\boldsymbol{\sigma} = \tilde{\mathbf{C}} \boldsymbol{\epsilon}\quad (2.5)$$

On the basis of Eq.s 2.2, the previous equation becomes

$$\begin{aligned}\sigma_p &= \tilde{\mathbf{C}}_{pp} \epsilon_p + \tilde{\mathbf{C}}_{pn} \epsilon_n \\ \sigma_n &= \tilde{\mathbf{C}}_{np} \epsilon_p + \tilde{\mathbf{C}}_{nn} \epsilon_n\end{aligned}\quad (2.6)$$

Both laminated box beams and cylinders can be considered constituted by a certain number of either straight or curved plates of orthotropic material, whose material coordinate systems (1,2,3) generally do not coincide with the physical coordinate system (x,y,z) (see Fig.2.2). Using this approach, the matrices of material coefficients of the generic material  $k$  are:

$$\tilde{\mathbf{C}}_{pp}^k = \begin{bmatrix} \tilde{C}_{11}^k & \tilde{C}_{12}^k & \tilde{C}_{14}^k \\ \tilde{C}_{12}^k & \tilde{C}_{22}^k & \tilde{C}_{24}^k \\ \tilde{C}_{14}^k & \tilde{C}_{24}^k & \tilde{C}_{44}^k \end{bmatrix}, \quad \tilde{\mathbf{C}}_{pn}^k = \begin{bmatrix} \tilde{C}_{15}^k & \tilde{C}_{16}^k & \tilde{C}_{13}^k \\ \tilde{C}_{25}^k & \tilde{C}_{26}^k & \tilde{C}_{23}^k \\ \tilde{C}_{45}^k & \tilde{C}_{46}^k & \tilde{C}_{43}^k \end{bmatrix}, \quad \tilde{\mathbf{C}}_{nn}^k = \begin{bmatrix} \tilde{C}_{55}^k & \tilde{C}_{56}^k & \tilde{C}_{35}^k \\ \tilde{C}_{56}^k & \tilde{C}_{66}^k & \tilde{C}_{36}^k \\ \tilde{C}_{35}^k & \tilde{C}_{36}^k & \tilde{C}_{33}^k \end{bmatrix}\quad (2.7)$$

On the other hand, for the FG materials (FGM), the matrices of material coefficients are

$$\tilde{\mathbf{C}}_{pp}^k = \begin{bmatrix} \tilde{C}_{11}^k & \tilde{C}_{12}^k & 0 \\ \tilde{C}_{12}^k & \tilde{C}_{22}^k & 0 \\ 0 & 0 & \tilde{C}_{66}^k \end{bmatrix}, \quad \tilde{\mathbf{C}}_{pn}^k = \begin{bmatrix} 0 & 0 & \tilde{C}_{13}^k \\ 0 & 0 & \tilde{C}_{23}^k \\ 0 & 0 & 0 \end{bmatrix}, \quad \tilde{\mathbf{C}}_{nn}^k = \begin{bmatrix} \tilde{C}_{55}^k & 0 & 0 \\ 0 & \tilde{C}_{44}^k & 0 \\ 0 & 0 & \tilde{C}_{33}^k \end{bmatrix} \quad (2.8)$$

where the Young's modulus ( $E$ ) and Poisson's ratio ( $\nu$ ) depend on the coordinates ( $x, y, z$ ) of the considered point.

For the sake of completeness, the explicit forms of the coefficients of the matrices  $\tilde{\mathbf{C}}$  are reported in Appendix A.

Models with constant and linear distributions of the in-plane displacement components,  $u_x$  and  $u_z$ , require modified material coefficients to overcome Poisson locking (see (Carrera et al., 2011a)). The correction process is based on the assumption of null transverse normal stresses,  $\sigma_{xx}$  and  $\sigma_{zz}$ , while the corresponding strain components,  $\epsilon_{xx}$  and  $\epsilon_{zz}$ , are accordingly calculated and substituted in the  $\sigma_{yy}$  and  $\sigma_{xy}$  expressions. These coefficients are used in the aforementioned linear cases. A more comprehensive analysis of the effect of Poisson locking and its correction can be found in Carrera et al. (2010b) and Carrera and Petrolo (2011b).

## 2.2 Carrera unified formulation

The unified formulation of the beam cross-section displacement field is described by an expansion of generic functions

$$\mathbf{u}(x, y, z, t) = F_\tau(x, z) \mathbf{u}_\tau(y, t), \quad \tau = 1, 2, \dots, M \quad (2.9)$$

where  $F_\tau(x, z)$  are functions of the cross-section coordinates  $x$  and  $z$ ,  $\mathbf{u}_\tau(y, t)$  is the displacement vector, and  $M$  stands for the number of terms of the expansion. According to the generalized Einstein notation, the repeated subscript  $\tau$  indicates summation. The choice of  $F_\tau$  and  $M$  is arbitrary, meaning that different base functions of any-order can be taken into account to model the kinematic field of a beam above the cross-section.

### 2.2.1 Equivalent-Single Layer Approach: ESL

The ESL approach provides a homogenization of the cross-sectional properties of the structure through summation of the contributions of each constituent part. This process yields an unique set of variables for the whole structure. Following this procedure, a number of different displacement theories has been considered in this work and some examples are outlined below.

#### Taylor-like expansions: TE 1D class

The first class considered uses Taylor-like expansions, which are obtained by means of 2D polynomials  $x^i z^j$  ( $i$  and  $j$  are positive integers). These models have been used in many applications and their capabilities have been widely discussed in the literature. For sake of completeness, Eqs. 2.10 and 2.11 show the TE models of second (TE2) and third (TE3) order, respectively

$$\begin{aligned} u_x &= u_{x1} + x u_{x2} + z u_{x3} + x^2 u_{x4} + xz u_{x5} + z^2 u_{x6} \\ u_y &= u_{y1} + x u_{y2} + z u_{y3} + x^2 u_{y4} + xz u_{y5} + z^2 u_{y6} \\ u_z &= u_{z1} + x u_{z2} + z u_{z3} + x^2 u_{z4} + xz u_{z5} + z^2 u_{z6} \end{aligned} \quad (2.10)$$



$$\begin{aligned}
u_x &= u_{x_1} + x u_{x_2} + z u_{x_3} + x^2 u_{x_4} + xz u_{x_5} + z^2 u_{x_6} + x^3 u_{x_7} + x^2 z u_{x_8} + xz^2 u_{x_9} + z^3 u_{x_{10}} \\
u_y &= u_{y_1} + x u_{y_2} + z u_{y_3} + x^2 u_{y_4} + xz u_{y_5} + z^2 u_{y_6} + x^3 u_{y_7} + x^2 z u_{y_8} + xz^2 u_{y_9} + z^3 u_{y_{10}} \\
u_z &= u_{z_1} + x u_{z_2} + z u_{z_3} + x^2 u_{z_4} + xz u_{z_5} + z^2 u_{z_6} + x^3 u_{z_7} + x^2 z u_{z_8} + xz^2 u_{z_9} + z^3 u_{z_{10}}
\end{aligned} \tag{2.11}$$

Regarding Eq.2.11, the refined expansion has 27 generalized displacement variables: three constant, six linear, nine parabolic and twelve cubic terms. The classical beam theories can be derived as particular cases of the first order Taylor-like expansion (TE1). Indeed, to obtain the first-order shear deformation theory (TBT) of Eq.2.12, the linear terms,  $u_{x_2}$ ,  $u_{x_3}$ ,  $u_{z_2}$ ,  $u_{z_3}$  can be neglected, and  $\theta(y)$  and  $\phi(y)$  are assumed equal to  $u_{y_2}$  and  $u_{y_3}$ . On the other hand, the Euler-Bernoulli (EBBT) beam model requires an additional condition resulting in the penalization of  $\epsilon_{xy}$  and  $\epsilon_{zy}$ . This condition can be imposed using a penalty technique in the constitutive equations.

$$\begin{aligned}
u_x &= u_{0x}(y) \\
u_y &= u_{0y}(y) + \theta(y)x + \phi(y)z \\
u_z &= u_{0z}(y)
\end{aligned} \tag{2.12}$$

### Advanced Theories based on Trigonometric and Exponential Expansions

Other ESL theories are obtained adopting trigonometric and exponential functions. In these cases, the displacement fields become

$$\begin{aligned}
u_x &= \sin\left(\frac{\pi x}{a}\right) u_{x_1} + \cos\left(\frac{\pi x}{a}\right) u_{x_2} + \sin\left(\frac{\pi z}{b}\right) u_{x_3} + \cos\left(\frac{\pi z}{b}\right) u_{x_4} + \sin\left(2\frac{\pi x}{a}\right) u_{x_5} + \dots \\
u_y &= \sin\left(\frac{\pi x}{a}\right) u_{y_1} + \sin\left(\frac{\pi x}{a}\right) u_{y_2} + \sin\left(\frac{\pi z}{b}\right) u_{y_3} + \cos\left(\frac{\pi z}{b}\right) u_{y_4} + \sin\left(2\frac{\pi x}{a}\right) u_{y_5} + \dots \\
u_z &= \sin\left(\frac{\pi x}{a}\right) u_{z_1} + \sin\left(\frac{\pi x}{a}\right) u_{z_2} + \sin\left(\frac{\pi z}{b}\right) u_{z_3} + \cos\left(\frac{\pi z}{b}\right) u_{z_4} + \sin\left(2\frac{\pi x}{a}\right) u_{z_5} + \dots
\end{aligned}$$

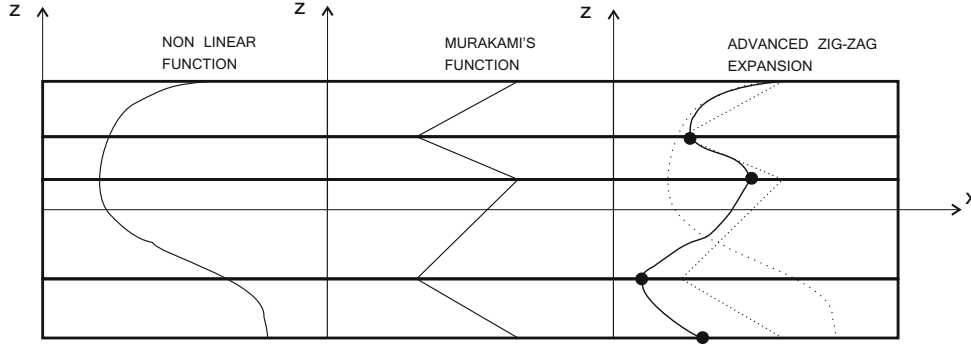
and

$$\begin{aligned}
u_x &= e^{\left(\frac{\pi}{a}\right)} u_{x_1} + e^{\left(\frac{\pi}{b}\right)} u_{x_2} + e^{\left(2\frac{\pi}{a}\right)} u_{x_3} + e^{\left(2\frac{\pi}{b}\right)} u_{x_4} + \dots \\
u_y &= e^{\left(\frac{\pi}{a}\right)} u_{y_1} + e^{\left(\frac{\pi}{b}\right)} u_{y_2} + e^{\left(2\frac{\pi}{a}\right)} u_{y_3} + e^{\left(2\frac{\pi}{b}\right)} u_{y_4} + \dots \\
u_z &= e^{\left(\frac{\pi}{a}\right)} u_{z_1} + e^{\left(\frac{\pi}{b}\right)} u_{z_2} + e^{\left(2\frac{\pi}{a}\right)} u_{z_3} + e^{\left(2\frac{\pi}{b}\right)} u_{z_4} + \dots
\end{aligned}$$

where  $a$ ,  $b$  are the main cross-section dimensions.

### Advanced zig-zag Theories

Murakami (1986) introduced his function in the first order shear deformation theory with the purpose of reproducing the zig-zag shape for the displacements of laminated plates. Due to the intrinsic anisotropy of multilayered structures, the first derivative of the displacement variables in the  $z$  – *direction* is discontinuous. The above function is able to describe the aforementioned discontinuity (see Fig.2.3).



**Figure 2.3** Physical and material reference system

Using the zig-zag function within the CUF framework, in the following, all theories containing this term are identified with the exponent ( $zz$ ). For example,  $TE2^{zz}$ :

$$\begin{aligned} u_x &= u_{x1} + x u_{x2} + z u_{x3} + x^2 u_{x4} + xz u_{x5} + z^2 u_{x6} + (-1)^k \zeta_k u_{x7Z} \\ u_y &= u_{y1} + x u_{y2} + z u_{y3} + x^2 u_{y4} + xz u_{y5} + z^2 u_{y6} + (-1)^k \zeta_k u_{y7Z} \\ u_z &= u_{z1} + x u_{z2} + z u_{z3} + x^2 u_{z4} + xz u_{z5} + z^2 u_{z6} + (-1)^k \zeta_k u_{z7Z} \end{aligned}$$

where  $\zeta_k = 2z_k/h_k$  is a non-dimensional layer coordinate and  $h_k$  the thickness of the  $k^{th}$ -layer. The exponent  $k$  changes the sign of the zig-zag term in each layer.

### Advanced Theories based on Miscellaneous Expansions

To further improve the kinematic model, we can combine the different functions, therefore obtaining miscellaneous expansions. One expansion is given below:

$$\begin{aligned} u_x &= u_{x1} + x u_{x2} + z u_{x3} + x^2 u_{x4} + xz u_{x5} + z^2 + e^{(1\frac{x}{a})} u_{x6} + e^{(1\frac{z}{b})} u_{x7} + \sin\left(3\frac{\pi x}{a}\right) u_{x8} \\ u_y &= u_{y1} + x u_{y2} + z u_{y3} + x^2 u_{y4} + xz u_{y5} + z^2 + e^{(1\frac{x}{a})} u_{y6} + e^{(1\frac{z}{b})} u_{y7} + \sin\left(3\frac{\pi x}{a}\right) u_{y8} \\ u_z &= u_{z1} + x u_{z2} + z u_{z3} + x^2 u_{z4} + xz u_{z5} + z^2 + e^{(1\frac{x}{a})} u_{z6} + e^{(1\frac{z}{b})} u_{z7} + \sin\left(3\frac{\pi x}{a}\right) u_{z8} \end{aligned}$$

Further cases will be directly described in the numerical discussion.

### 2.2.2 The Layer-Wise Approach: LW

According to the LW approach, a different set of variables may be assumed for each part that constitutes the structure. Therefore, the homogenization process is only conducted in correspondence to the shared interfaces between two cross-sectional regions. Although the Lagrange 1D class can be also used in the ESL framework, its main feature is the capability to provide an accurate component-wise description (therefore layer-wise) of the structure.

#### Lagrange expansion (LE) 1D class

LE models use Lagrange polynomials to build 1D higher-order theories and the isoparametric

formulation is exploited to deal with arbitrary shaped geometries. For the nine-point element (L9) (see Fig.2.4), the interpolation functions are given by

$$\begin{aligned}
 F_\tau &= \frac{1}{4}(r^2 + r r_\tau)(s^2 + s s_\tau) & \tau = 1, 3, 5, 7 \\
 F_\tau &= \frac{1}{2}s_\tau^2(s^2 - s s_\tau)(1 - r^2) + \frac{1}{2}r_\tau^2(r^2 - r r_\tau)(1 - s^2) & \tau = 2, 4, 6, 8 \\
 F_\tau &= (1 - r^2)(1 - s^2) & \tau = 9
 \end{aligned} \tag{2.13}$$

where  $r$  and  $s$  range from  $-1$  to  $+1$ . The displacement field given by an L9 element is

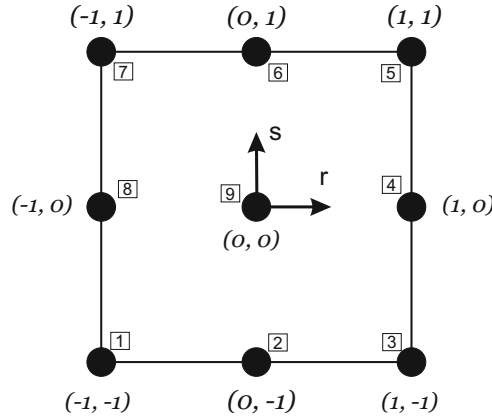


Figure 2.4 Cross-section L9 element

$$\begin{aligned}
 u_x &= F_1 u_{x1} + F_2 u_{x2} + F_3 u_{x3} + F_4 u_{x4} + F_5 u_{x5} + F_6 u_{x6} + F_7 u_{x7} + F_8 u_{x8} + F_9 u_{x9} \\
 u_y &= F_1 u_{y1} + F_2 u_{y2} + F_3 u_{y3} + F_4 u_{y4} + F_5 u_{y5} + F_6 u_{y6} + F_7 u_{y7} + F_8 u_{y8} + F_9 u_{y9} \\
 u_z &= F_1 u_{z1} + F_2 u_{z2} + F_3 u_{z3} + F_4 u_{z4} + F_5 u_{z5} + F_6 u_{z6} + F_7 u_{z7} + F_8 u_{z8} + F_9 u_{z9}
 \end{aligned} \tag{2.14}$$

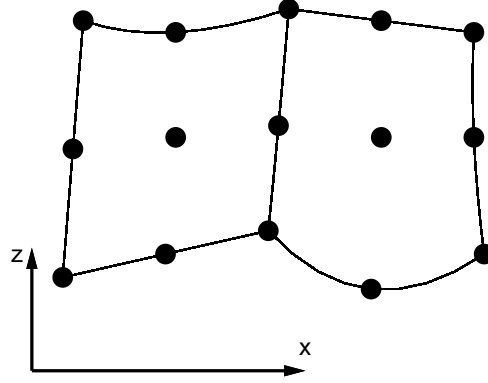
where  $u_{x1}, \dots, u_{z9}$  are the displacement variables of the problem and they represent the translational displacement components of each of the nine points of the L9 element. This means that LE models provide elements that only have displacement variables. According to Carrera and Petrolo (2011b), the beam cross-section can be discretized using several L-elements for extra refinements, as shown in Fig.2.5 where two L9 elements are assembled.

For further details on Lagrange 1D models, we refer to (Carrera and Petrolo, 2011b,c) where L3, L4, L6 and L16 sets are also described.

## 2.3 The Finite Element formulation

The governing equations of a system can be derived by means of different variational principles of Mechanics such as the Principle of Virtual Displacements (PVD), D'Alembert's Principle and Hamilton's Principle. Regardless of the chosen procedure, with the use of CUF, all contributions in the governing equations (mass, stiffness, damping and loading arrays) are written in terms of '*fundamental nuclei*', whose forms are independent of the order of models. In order to overcome the limitations of the analytical approaches, the weak form of the governing equations is solved using the Finite Element method (FE). The generalized displacement vector (see Eq.2.9) is given by :

$$\mathbf{u}_\tau(y, t) = N_i(y) \mathbf{q}_{\tau i}(t) \tag{2.15}$$



**Figure 2.5** Two assembled L9 elements in actual geometry.

where  $N_i$  are the shape functions and  $\mathbf{q}_{\tau i}(t)$  is the nodal displacement vector:

$$\mathbf{q}_{\tau i}(t) = \{ q_{u_{x\tau i}} \quad q_{u_{y\tau i}} \quad q_{u_{z\tau i}} \}^T \quad i = 1, \dots, N_{node} \quad (2.16)$$

with ' $N_{node}$ ' as the number of nodes of the element. Hereafter, the vector dependencies on time (t) and spatial coordinates (x,y,z) will be omitted. Here three different finite elements are considered to provide a linear (B2), a quadratic (B3) and a cubic (B4) interpolation of the displacement variable along the beam axis. The lagrangian shape functions of the B2, B3 and B4 elements are thus defined as,

$$\begin{aligned} N_1 &= \frac{1}{2}(1-r), \quad N_2 = \frac{1}{2}(1+r), \quad \begin{cases} r_1 = -1 \\ r_2 = +1 \end{cases} \\ N_1 &= \frac{1}{2}r(r-1), \quad N_2 = \frac{1}{2}r(r+1), \quad N_3 = -(1+r)(1-r), \quad \begin{cases} r_1 = -1 \\ r_2 = +1 \\ r_3 = 0 \end{cases} \\ N_1 &= -\frac{9}{16}(r + \frac{1}{3})(r - \frac{1}{3})(r-1), \quad N_2 = \frac{9}{16}(r + \frac{1}{3})(r - \frac{1}{3})(r+1), \quad \begin{cases} r_1 = -1 \\ r_2 = +1 \\ r_3 = -\frac{1}{3} \\ r_4 = +\frac{1}{3} \end{cases} \\ N_3 &= +\frac{27}{16}(r+1)(r - \frac{1}{3})(r-1), \quad N_4 = -\frac{27}{16}(r+1)(r + \frac{1}{3})(r-1), \end{aligned} \quad (2.17)$$

where the natural coordinate,  $r$ , varies from  $-1$  to  $+1$  and  $r_i$  indicates the position of the node within the natural beam boundaries.



## Chapter 3

# Carrera Unified Formulation in Rotordynamics

*In this chapter, the equations of motion (EoM) for rotating structures in CUF form are derived by means of Hamilton's Principle. Then, the fundamental nucleus expression of each term in the equations are provided.*

### 3.1 Hamilton's Principle and Lagrange's Equations

The general form of Hamilton's Principle states that the variation of the kinetic and potential energy and the variation of the work exerted by non-conservative forces must equal zero during any time interval. It may be expressed as

$$\int_{t_0}^{t_1} \delta (T - U) dt + \int_{t_0}^{t_1} \delta W_{nc} dt = 0 \quad (3.1)$$

where  $\delta$  refers to kinematically admissible perturbations during the indicated time interval and

- T: Total Kinetic Energy of the system
- U: Total Potential Energy of the system, including both strain energy and potential of any conservative external forces
- $W_{nc}$ : Work done by non-conservative forces acting on the system, including any arbitrary external loads

The application of this variational concept leads directly to the EoM for any given system. If we express the energies in terms of generalised coordinates

$$T = T(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n, t) \quad U = U(q_1, \dots, q_n, t) \quad \delta W_{nc} = Q_1 \delta q_1 + \dots + Q_n \delta q_n \quad (3.2)$$

where ' $Q_k$ ' are the generalised forces, the energy variations can be presented as

$$\delta T = \sum_{k=1}^n \left( \frac{\partial T}{\partial q_k} \delta q_k + \frac{\partial T}{\partial \dot{q}_k} \delta \dot{q}_k \right) \quad \delta U = \sum_{k=1}^n \left( \frac{\partial U}{\partial q_k} \delta q_k \right) \quad \delta W_{nc} = \sum_{k=1}^n (Q_k \delta q_k) \quad (3.3)$$

Substituting Eqs.3.3 into Eq.3.1, we obtain

$$\int_{t_1}^{t_2} \left[ \sum_{k=1}^n \left( \frac{\partial T}{\partial q_k} \delta q_k + \frac{\partial T}{\partial \dot{q}_k} \delta \dot{q}_k \right) - \sum_{k=1}^n \left( \frac{\partial U}{\partial q_k} \delta q_k \right) + \sum_{k=1}^n (Q_k \delta q_k) \right] \quad (3.4)$$

Integration by parts on the time-dependent terms yields

$$\int_{t_1}^{t_2} \left[ \sum_{k=1}^n \left( \frac{\partial T}{\partial q_k} - \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_k} \right) - \frac{\partial U}{\partial q_k} + Q_k \right) \delta q_k \right] dt = 0 \quad (3.5)$$

Finally, since the variation must be arbitrary and independent, we obtain Lagrange's equations that directly lead to EoM.

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_k} \right) - \frac{\partial T}{\partial q_k} + \frac{\partial U}{\partial q_k} = Q_k \quad (3.6)$$

The solutions of Eq.3.6 are the stationary points of Eq.3.1.

### 3.1.1 The structural stiffness matrix and the loading vector in CUF form

In order to derive the expressions of stiffness matrix and loading vector, we consider a structure subjected to a generic concentrate load  $\mathbf{P}$  applied at  $(x^p, y^p, z^p)$ , while its kinetic energy is assumed null.

$$\mathbf{P} = \{P_x \quad P_y \quad P_z\}^T \quad (3.7)$$

The potential energy can be expressed as

$$U = \frac{1}{2} \int_V (\mathbf{u}^T \mathbf{D}^T \mathbf{C} \mathbf{D} \mathbf{u}) dV \quad (3.8)$$

where  $\mathbf{u}$  is the displacement vector of Eq.2.1,  $\mathbf{D}$  is the matrix of linear differential operators and  $\mathbf{C}$  is the matrix of material coefficients. Substituting Eqs.3.7 and 3.8 in Eq.3.1 we obtain

$$\int_{t_0}^{t_1} \left( \int_V \delta \mathbf{u}^T (\mathbf{D}^T \mathbf{C} \mathbf{D}) \mathbf{u} dV - \delta \mathbf{u}^T \mathbf{P} \right) dt = 0 \quad (3.9)$$

Then, introducing the CUF (Eq2.9) and FE (Eq.2.15) approximations, Eq.3.9 becomes

$$\int_{t_0}^{t_1} \left( \delta \mathbf{q}_{\tau i}^T \mathbf{K}^{ij\tau s} \mathbf{q}_{sj} - \delta \mathbf{q}_{\tau i}^T \tilde{\mathbf{P}}^{i\tau} \right) dt = 0 \quad (3.10)$$

where  $\mathbf{K}^{ij\tau s}$  and  $\tilde{\mathbf{P}}^{i\tau}$  are the structural stiffness matrix and the loading vector in terms of the fundamental nuclei, whose expressions are reported below

$$\begin{aligned}
\mathbf{K}^{ij\tau s} = & I_l^{ij} \triangleleft \mathbf{D}_{np}^T(F_\tau \mathbf{I}) \left[ \tilde{\mathbf{C}}_{np}^k \mathbf{D}_p(F_s \mathbf{I}) + \tilde{\mathbf{C}}_{nn}^k \mathbf{D}_{np}(F_s \mathbf{I}) \right] + \\
& + \mathbf{D}_p^T(F_\tau \mathbf{I}) \left[ \tilde{\mathbf{C}}_{pp}^k \mathbf{D}_p(F_s \mathbf{I}) + \tilde{\mathbf{C}}_{pn}^k \mathbf{D}_{np}(F_s \mathbf{I}) \right] \triangleright + \\
& + I_l^{ij,y} \triangleleft \left[ \mathbf{D}_{np}^T(F_\tau \mathbf{I}) + \mathbf{D}_p^T(F_\tau \mathbf{I}) \tilde{\mathbf{C}}_{pn}^k \right] F_s \triangleright + \mathbf{I}_{Ay} \\
& + I_l^{i,yj} \mathbf{I}_{Ay}^T \triangleleft F_\tau \left[ \tilde{\mathbf{C}}_{np}^k \mathbf{D}_p(F_s \mathbf{I}) + \tilde{\mathbf{C}}_{nn}^k \mathbf{D}_{np}(F_s \mathbf{I}) \right] \triangleright + \\
& + I_l^{i,yj,y} \mathbf{I}_{Ay}^T \mathbf{I}_{Ay} \triangleleft F_\tau \tilde{\mathbf{C}}_{nn}^k F_s \triangleright
\end{aligned} \tag{3.11}$$

$$\tilde{\mathbf{P}}^{i\tau} = N_i(y^p) F_\tau(x^p, z^p) \mathbf{P}$$

where

$$\mathbf{I}_{Ay} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \triangleleft \dots \triangleright = \int_A \dots dA \tag{3.12}$$

$$\left( I_l^{ij}, I_l^{ij,y}, I_l^{i,yj}, I_l^{i,yj,y} \right) = \int_l \left( N_i N_j, N_i N_{j,y}, N_{i,y} N_j, N_{i,y} N_{j,y} \right) dy \tag{3.13}$$

For the sake of completeness, in Appendix B, the nine components of the fundamental nucleus of the matrix  $\mathbf{K}^{ij\tau s}$  are written in an explicit form. We refer to (Carrera et al., 2014b) for a detailed discussion on the computation of loading vector. It must be noted that

1.  $\mathbf{K}^{ij\tau s}$  does not depend on the expansion order.
2.  $\mathbf{K}^{ij\tau s}$  does not depend on the choice of the  $F_\tau$  expansion polynomials.

These are the key-points of CUF which permits, with only nine statements, to implement any-order of multiple class theories.

After a classical assembly procedure, which has been illustrated in Sec.2.3, and applying Eq.3.6 we obtain the following algebraic problem

$$\tilde{\mathbf{K}} \mathbf{q} = \tilde{\mathbf{P}} \tag{3.14}$$

where  $\tilde{\mathbf{k}}$ ,  $\tilde{\mathbf{P}}$  and  $\mathbf{q}$  indicate the global stiffness matrix, the loading and the unknown vectors, respectively.

## 3.2 Dynamics of spinning structures

When a structure is rotating, inertial forces and moments are observed. We consider a beam that is free to rotate about its longitudinal axis with a constant spin-speed  $\Omega$  and supported by  $N_b$  bearings. In this case, Hamilton's Principle can be expressed as

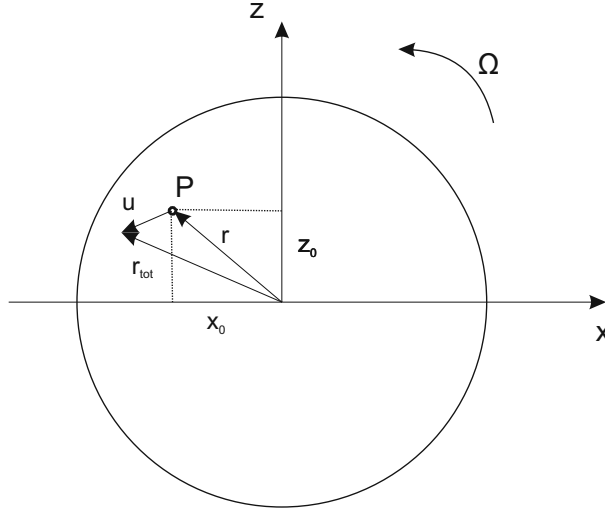
$$\delta \int_{t_0}^{t_1} (T - (U + U_{\sigma_0}) + W_b) dt = 0 \tag{3.15}$$

where the terms ' $W_b$ ' and ' $U_{\sigma_0}$ ' are the contributions due to the bearings and centrifugal effect, respectively. Following the procedure described in Sec.3.1.1, we derive the variations of the different energetic contributions.



### 3.2.1 The Kinetic Energy

Figure 3.1 shows the rotating reference frame and the cross-section of the beam.



**Figure 3.1** Sketch of the rotating reference frame.

The absolute velocity of the point  $P$  is the sum of relative and transfer velocities.

$$\mathbf{v}_{abs} = \mathbf{v}_{rel} + \mathbf{v}_{tr} = \dot{\mathbf{u}} + \boldsymbol{\Omega} \times \mathbf{r}_{tot} \quad (3.16)$$

$$\boldsymbol{\Omega} = \begin{bmatrix} 0 & 0 & \Omega \\ 0 & 0 & 0 \\ -\Omega & 0 & 0 \end{bmatrix}$$

where  $\mathbf{r}_{tot} = \mathbf{r} + \mathbf{u}$  is the distance of  $P$  from the neutral axis. Hence, the kinetic energy of the whole structure is

$$T = \frac{1}{2} \int_V \rho (\dot{\mathbf{u}}^T \dot{\mathbf{u}} + 2\mathbf{u}^T \boldsymbol{\Omega}^T \dot{\mathbf{u}} + \mathbf{u}^T \boldsymbol{\Omega}^T \boldsymbol{\Omega} \mathbf{u} + 2\dot{\mathbf{u}}^T \boldsymbol{\Omega} \mathbf{r} + 2\mathbf{u}^T \boldsymbol{\Omega}^T \boldsymbol{\Omega} \mathbf{r} + \mathbf{r}^T \boldsymbol{\Omega}^T \boldsymbol{\Omega} \mathbf{r}) dV \quad (3.17)$$

### 3.2.2 The Centrifugal Potential Energy

The geometrical potential energy,  $U_{\sigma_0}$ , is fundamental when a highly deformable structure is rotating.

$$U_{\sigma_0} = \int_V (\boldsymbol{\sigma}_0^T \boldsymbol{\epsilon}_{nl}) dV \quad (3.18)$$

This term depends on non-linear term components of the strain field  $\boldsymbol{\epsilon}_{nl}$  and the pre-stress  $\boldsymbol{\sigma}_0$ . The  $\boldsymbol{\sigma}_0$  distribution is function of the structure characteristics and it is therefore, changed according to the studied problem.

The pre-stress field for axial-symmetric structures is typically provided in terms of circumferential ( $\sigma_{\delta\delta}$ ), radial ( $\sigma_{rr}$ ) and axial ( $\sigma_{yy}$ ) contributions that, written in a cartesian coordinate system, are

$$\begin{aligned}
\sigma_{0xx} &= \sigma_{\delta\delta} \sin(\delta)^2 + \sigma_{rr} \cos(\delta)^2 \\
\sigma_{0yy} &= \sigma_{yy} \\
\sigma_{0zz} &= \sigma_{\delta\delta} \cos(\delta)^2 + \sigma_{rr} \sin(\delta)^2 \\
\sigma_{0xz} (= \sigma_{0zx}) &= \sigma_{rr} \sin(\delta) \cos(\delta) - \sigma_{\delta\delta} \sin(\delta) \cos(\delta)
\end{aligned}$$

In the case of a thin-walled cylinder, the only initial stress acts in the circumferential direction and it is assumed equal to

$$\sigma_0 = \rho \Omega^2 R_m, \quad (3.19)$$

where ' $\rho$ ' is the density and ' $R_m$ ' is the mean radius. Conversely, for a deformable blade, the centrifugal pre-stress acts along the radial direction and its expression is

$$\sigma_0 = \Omega^2 \rho \left[ r_h R + \frac{1}{2} R^2 - r_h r - \frac{1}{2} r^2 \right] \quad (3.20)$$

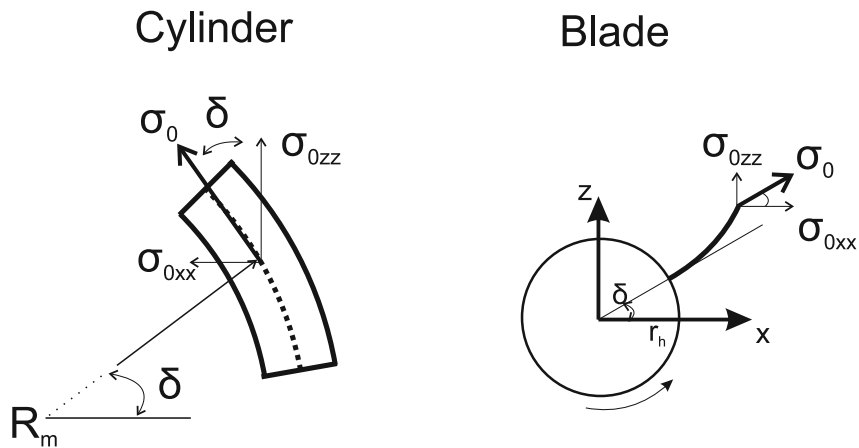
where ' $R$ ' is the maximum distance from the hub, whose radius is ' $r_h$ '. Regarding a thin hollow disk, the radial ' $\sigma_r^\Omega$ ' and circumferential ' $\sigma_c^\Omega$ ' contributions are defined by Manson's equations

$$\begin{aligned}
\sigma_r^\Omega &= \sigma_0 \frac{(3+\nu)}{8} \left[ 1 + \frac{D_i^2}{D_e^2} - \frac{D_i^2}{D^2} - \frac{D_e^2}{D^2} \right] \\
\sigma_c^\Omega &= \sigma_0 \frac{(3+\nu)}{8} \left[ 1 + \frac{D_i^2}{D_e^2} + \frac{D_i^2}{D^2} - \frac{(1+3\nu)}{(3+\nu)} \frac{D_e^2}{D^2} \right]
\end{aligned} \quad (3.21)$$

in which ' $D_i$ ' and ' $D_e$ ' are the inner and external diameters of the disk, respectively. The term  $\sigma_0$  coincides with Eq.3.19.

Regardless of the considered structure, the geometrical potential energy is defined as

$$U_{\sigma_0} = \int_V \sigma_{0xx} \epsilon_{xx}^{nl} + \sigma_{0zz} \epsilon_{zz}^{nl} + \sigma_{0xz} \epsilon_{xz}^{nl} + \sigma_{0zx} \epsilon_{zx}^{nl} dV \quad (3.22)$$



**Figure 3.2** The centrifugal stiffness.

### 3.2.3 Stiffness and Damping due to bearings

The  $N_b$  bearings are modeled as springs and viscous dampers, therefore, the term  $W_b$  is expressed as

$$W_b = \sum_{p=1}^{N_b} \mathbf{u}^T \mathbf{K}_b^p \mathbf{u} + \dot{\mathbf{u}}^T \mathbf{C}_b^p \mathbf{u} \quad (3.23)$$

Considering the generic  $p$ -th bearing whose coordinates are  $(x_b, y_b, z_b)$ , the stiffness and damping coefficients are

$$\mathbf{K}_b^p = \begin{bmatrix} k_{xx}^p & k_{xy}^p & k_{xz}^p \\ k_{yx}^p & k_{yy}^p & k_{yz}^p \\ k_{zx}^p & k_{zy}^p & k_{zz}^p \end{bmatrix}, \quad \mathbf{C}_b^p = \begin{bmatrix} c_{xx}^p & c_{xy}^p & c_{xz}^p \\ c_{yx}^p & c_{yy}^p & c_{yz}^p \\ c_{zx}^p & c_{zy}^p & c_{zz}^p \end{bmatrix} \quad (3.24)$$

### 3.2.4 The Equations of Motion

Similarly to the static case of Sec.3.1.1, introducing the CUF (Eq.2.9) and FE (Eq.2.15) approximations and substituting Eqs.3.8, 3.17, 3.22 and 3.23 in Eq.3.15, we obtain

$$\int_{t_0}^{t_1} \left( \delta \mathbf{q}_{\tau i}^T \mathbf{M}^{ij\tau s} \ddot{\mathbf{q}}_{sj} + \delta \mathbf{q}_{\tau i}^T \mathbf{G}_{TOT}^{ij\tau s} \dot{\mathbf{q}}_{sj} + \delta \mathbf{q}_{\tau i}^T \mathbf{K}_{TOT}^{ij\tau s} \mathbf{q}_{sj} + \delta \mathbf{q}_{\tau i}^T \mathbf{F}_{\Omega}^{i\tau} \mathbf{r} \right) dt = 0 \quad (3.25)$$

where

$$\begin{aligned} \mathbf{G}_{TOT}^{ij\tau s} &= \mathbf{G}_{\Omega}^{ij\tau s} + \mathbf{G}_b^{ij\tau s} \\ \mathbf{K}_{TOT}^{ij\tau s} &= \mathbf{K}^{ij\tau s} + \mathbf{K}_b^{ij\tau s} + \mathbf{K}_{\Omega}^{ij\tau s} + \mathbf{K}_{\sigma_0}^{ij\tau s} \end{aligned} \quad (3.26)$$

In addition to the mass matrix  $\mathbf{M}^{ij\tau s}$  and the stiffness matrix  $\mathbf{K}^{ij\tau s}$ , the rotation introduces the following terms

- the Coriolis matrix  $\mathbf{G}^{ij\tau s}$
- the matrix of bearing damping  $\mathbf{G}_b^{ij\tau s}$
- the matrix of bearing stiffness  $\mathbf{K}_b^{ij\tau s}$
- the matrix of spin softening  $\mathbf{K}_{\Omega}^{ij\tau s}$
- the matrix of centrifugal stiffening  $\mathbf{K}_{\sigma_0}^{ij\tau s}$
- the load vector  $\mathbf{F}_{\Omega}^{i\tau}$

As previously done for the stiffness matrix  $\mathbf{K}^{ij\tau s}$ , the *fundamental nuclei* related to the above matrices are reported below

$$\begin{aligned}
M^{ij\tau s} &= I_l^{ij} \triangleleft (F_\tau \rho^k \mathbf{I} F_s) \triangleright \\
\mathbf{G}_\Omega^{ij\tau s} &= I_l^{ij} \triangleleft (F_\tau \rho^k \mathbf{I} F_s) \triangleright 2\Omega \\
\mathbf{G}_b^{ij\tau s} &= \sum_{p=1}^{N_b} N_i(y_b^p) F_\tau(x_b^p, z_b^p) C_b^p F_s(x_b^p, z_b^p) N_j(y_b^p) \\
\mathbf{K}_b^{ij\tau s} &= \sum_{p=1}^{N_b} N_i(y_b^p) F_\tau(x_b^p, z_b^p) \mathbf{K}_b^p F_s(x_b^p, z_b^p) N_j(y_b^p) \\
\mathbf{K}_\Omega^{ij\tau s} &= -I_l^{ij} \triangleleft (F_\tau \rho^k \mathbf{I} F_s) \triangleright \Omega^T \Omega \\
\mathbf{K}_{\sigma_0}^{ij\tau s} &= I_l^{ij} \triangleleft (F_{\tau,x} \sigma_{0xx} \mathbf{I} F_{s,x}) + (F_{\tau,z} \sigma_{0zz} \mathbf{I} F_{s,z}) + \\
&\quad + (F_{\tau,x} \sigma_{0xz} \mathbf{I} F_{s,z}) + (F_{\tau,z} \sigma_{0zx} \mathbf{I} F_{s,x}) \triangleright \\
\mathbf{F}_\Omega^{i\tau} &= I_l^i \Omega^T \Omega \triangleleft F_\tau \mathbf{I} \rho \mathbf{r} \triangleright
\end{aligned} \tag{3.27}$$

where

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.28}$$

$$\triangleleft \dots \triangleright = \int_A \dots dA \tag{3.29}$$

$$\left( I_l^i, I_l^{ij}, I_l^{ij,y}, I_l^{i,yj}, I_l^{i,yj,y} \right) = \int_l \left( N_i, N_i N_j, N_i N_{j,y}, N_{i,y} N_j, N_{i,y} N_{j,y} \right) dy \tag{3.30}$$

and  $\mathbf{r} = \{x_P, 0, z_P\}^T$  is the distance of a generic point  $P$  on the cross-section from the neutral axis. In matrix form, the new *fundamental nuclei* are

$$\begin{aligned}
\mathbf{G}_\Omega^{ij\tau s} &= \Omega \begin{bmatrix} 0 & 0 & -2 I_l^{i,j} \triangleleft F_\tau \rho F_s \triangleright \\ 0 & 0 & 0 \\ 2 I_l^{i,j} \triangleleft F_\tau \rho F_s \triangleright & 0 & 0 \end{bmatrix} \\
\mathbf{K}_\Omega^{ij\tau s} &= \Omega^2 \begin{bmatrix} -I_l^{i,j} \triangleleft F_\tau \rho F_s \triangleright & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -I_l^{i,j} \triangleleft F_\tau \rho F_s \triangleright \end{bmatrix} \\
\mathbf{K}_{\sigma_0}^{ij\tau s} &= \Omega^2 \begin{bmatrix} I_l^{ij} K_{\sigma_0} & 0 & 0 \\ 0 & I_l^{ij} (K_{\sigma_0}) & 0 \\ 0 & 0 & I_l^{ij} K_{\sigma_0} \end{bmatrix} \\
\mathbf{F}_\Omega^{i\tau} &= \Omega^2 \begin{bmatrix} I_l^i \triangleleft F_\tau \rho x_P \triangleright \\ 0 \\ I_l^i \triangleleft F_\tau \rho z_P \triangleright \end{bmatrix}
\end{aligned} \tag{3.31}$$

To obtain the natural frequencies and the normal modes of the rotor, we solve the homogeneous equation:

$$\mathbf{M} \ddot{\mathbf{q}} + \mathbf{G}_{TOT} \dot{\mathbf{q}} + \mathbf{K}_{TOT} \mathbf{q} = 0 \quad (3.32)$$

Assuming a periodic function as solution

$$\mathbf{q} = \bar{\mathbf{q}} e^{i\omega t} \quad (3.33)$$

Substituting Eq.3.33 and its derivative in Eq.3.32, we obtain

$$\bar{\mathbf{q}} e^{i\omega t} [\mathbf{K}_{TOT} + \mathbf{G}_{TOT} i\omega - \mathbf{M} \omega^2] = 0 \quad (3.34)$$

The quadratic eigenvalues problem (QEP) of generic order  $R$  in Eq.3.34 is transformed in a classical linear system of order  $2 \times R$ :

$$\begin{cases} \mathbf{M} \ddot{\mathbf{q}} + \mathbf{G}_{TOT} \dot{\mathbf{q}} + \mathbf{K}_{TOT} \mathbf{q} = 0 \\ -\dot{\mathbf{q}} + \dot{\mathbf{q}} = 0 \end{cases} \quad (3.35)$$

Now, introducing:

$$\mathbf{a} = \begin{Bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{Bmatrix} \quad \dot{\mathbf{a}} = \begin{Bmatrix} \dot{\mathbf{q}} \\ \ddot{\mathbf{q}} \end{Bmatrix} \quad (3.36)$$

the equations of motion assume the form:

$$\frac{\mathbf{R}}{\mathbf{T}} - \frac{1}{i\omega} \mathbf{I} = 0 \quad (3.37)$$

where:

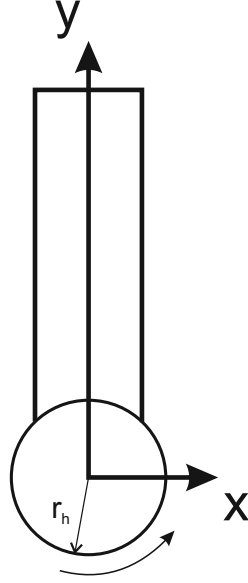
$$\mathbf{T}^{-1} \mathbf{R} = \begin{bmatrix} (\mathbf{K}_{TOT})^{-1}(\mathbf{G}_{TOT}) & (\mathbf{K}_{TOT})^{-1}(\mathbf{M}) \\ -\mathbf{I} & \mathbf{0} \end{bmatrix} \quad (3.38)$$

The problem in Eq.3.38 is in the classical form and can be solved with the standard eigensolvers.

### 3.3 Dynamics of centrifugally stiffened structures

We consider, in this section, a structure that is rotating about the z-axis as shown in Fig.3.3. In this case, the work done by supports is not taken into account, therefore, Hamilton's principle becomes

$$\delta \int_{t_0}^{t_1} (T - (U + U_{\sigma_0})) dt = 0 \quad (3.39)$$



**Figure 3.3** Sketch of a centrifugally stiffened structure.

### 3.3.1 The Kinetic Energy

The constant rotational speed matrix is

$$\mathbf{\Omega} = \begin{bmatrix} 0 & -\Omega & 0 \\ \Omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, the kinetic energy of the whole structure becomes

$$T = \frac{1}{2} \int_V \rho (\dot{\mathbf{u}}^T \dot{\mathbf{u}} + 2\mathbf{u}^T \mathbf{\Omega}^T \dot{\mathbf{u}} + \mathbf{u}^T \mathbf{\Omega}^T \mathbf{\Omega} \mathbf{u} + 2\dot{\mathbf{u}}^T \mathbf{\Omega} \mathbf{r} + 2\mathbf{u}^T \mathbf{\Omega}^T \mathbf{\Omega} \mathbf{r}) dV \quad (3.40)$$

### 3.3.2 The Centrifugal Potential Energy

Although the ' $\sigma_0$ ' distribution is the same of Eq.3.20, the expression of the centrifugal potential energy involves non-linear strain terms that are different from the previous case

$$U_{\sigma_0} = \frac{1}{2} \int_V \sigma_0 (u_{x,y}^2 + u_{y,y}^2 + u_{z,y}^2) dV \quad (3.41)$$

It should be noted that, despite the non-linear contribution ' $u_{y,y}$ ' is usually omitted in the literature, its role may be important due to the coupling between axial and chordwise motions.

### 3.3.3 The Equations of Motion

Once again introducing the CUF (Eq.2.9) and FE (Eq.2.15) approximations and substituting Eqs.3.8, 3.40 and 3.41 Eq.3.39, we obtain

$$\int_{t_0}^{t_1} \left( \delta \mathbf{q}_{\tau i}^T \mathbf{M}^{ij\tau s} \ddot{\mathbf{q}}_{sj} + \delta \mathbf{q}_{\tau i}^T \mathbf{G}_{TOT}^{ij\tau s} \dot{\mathbf{q}}_{sj} + \delta \mathbf{q}_{\tau i}^T \mathbf{K}_{TOT}^{ij\tau s} \mathbf{q}_{sj} + \delta \mathbf{q}_{\tau i}^T \mathbf{F}_{\Omega}^{i\tau} \mathbf{r} \right) dt = 0 \quad (3.42)$$

where

$$\mathbf{G}_{TOT}^{ij\tau s} = \mathbf{G}_{\Omega}^{ij\tau s} \quad (3.43)$$

$$\mathbf{K}_{TOT}^{ij\tau s} = \mathbf{K}^{ij\tau s} + \mathbf{K}_{\Omega}^{ij\tau s} + \mathbf{K}_{\sigma_0}^{ij\tau s}$$

The *fundamental nuclei*  $\mathbf{G}_{\Omega}^{ij\tau s}$  and  $\mathbf{K}_{\Omega}^{ij\tau s}$  have been already defined in Sec.3.2.4, while  $\mathbf{K}_{\sigma_0}^{ij\tau s}$  is reported below

$$\mathbf{K}_{\sigma_0}^{ij\tau s} = I_{l\sigma_0}^{i,yj,y} \triangleleft (F_{\tau} \rho \mathbf{I} F_s) \triangleright \Omega^T \Omega \quad (3.44)$$

$$\mathbf{F}_{\Omega}^{i\tau} = I_{ly}^i \Omega^T \Omega \triangleleft F_{\tau} \mathbf{I} \rho \triangleright$$

where

$$I_{ly}^i, I_{l\sigma_0}^{i,yj,y} = \int_l y N_i, \sigma_0 N_{i,y} N_{j,y} dy \quad (3.45)$$

In this case, the *fundamental nuclei* in matrix form yield

$$\mathbf{G}_{\Omega}^{ij\tau s} = \Omega \begin{bmatrix} 0 & -2 I_l^{i,j} \triangleleft F_{\tau} \rho F_s \triangleright & 0 \\ 2 I_l^{i,j} \triangleleft F_{\tau} \rho F_s \triangleright & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{K}_{\Omega}^{ij\tau s} = \Omega^2 \begin{bmatrix} -I_l^{i,j} \triangleleft F_{\tau} \rho F_s \triangleright & 0 & 0 \\ 0 & -I_l^{i,j} \triangleleft F_{\tau} \rho F_s \triangleright & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{K}_{\sigma_0}^{ij\tau s} = \Omega^2 \begin{bmatrix} I_{l\sigma_0}^{i,yj,y} \triangleleft (F_{\tau} \rho \mathbf{I} F_s) \triangleright & 0 & 0 \\ 0 & I_{l\sigma_0}^{i,yj,y} \triangleleft (F_{\tau} \rho \mathbf{I} F_s) \triangleright & 0 \\ 0 & 0 & I_{l\sigma_0}^{i,yj,y} \triangleleft (F_{\tau} \rho \mathbf{I} F_s) \triangleright \end{bmatrix} \quad (3.46)$$

$$\mathbf{F}_{\Omega}^{i\tau} = \Omega^2 \begin{bmatrix} 0 \\ I_{ly}^i \triangleleft F_{\tau} \rho \triangleright \\ 0 \end{bmatrix}$$

Following the procedure explained in Sec.3.2.4, we obtain the natural frequencies and normal modes solving Eq.3.38.

The proposed formulation allows to write EoM in a fully three-dimensional form for both spinning and centrifugally stiffened structures. Performance of the 1D-CUF elements in rotordynamics has been evaluated in the result section.

## Chapter 4

# Aeroelastic models

*In this chapter, the expressions of the external work exerted by aerodynamic loads are derived in CUF form. The aerodynamic loads are defined according to steady and unsteady theories for subsonic and supersonic flow regimes.*

### 4.1 2-D steady and unsteady aerodynamic theories for fixed-wing

Theodorsen was the first to propose a complete solution for a thin airfoil that undergoes harmonic lateral oscillations in an incompressible fluid. Adopting the assumptions of the small-disturbance theory, Theodorsen considered a flat plate with a control surface which could move in vertical translation  $h(t)$  and rotate about the axis at  $x = ba$  through by angle  $\alpha(t)$ . The final expression of the lift for a unit of span, due to both translation and rotation as derived in (Bisplinghoff et al., 1996), is

$$L = \pi\rho_a b^2 [\ddot{h} + U\dot{\alpha} - ba\ddot{\alpha}] + 2\pi\rho_a U b C(k) \left[ \dot{h} + U\alpha + b \left( \frac{1}{2} - a \right) \dot{\alpha} \right] \quad (4.1)$$

where ' $U$ ' is the free-stream velocity, ' $b$ ' the semichord, ' $\rho_a$ ' the air density and ' $C(k)$ ' the deficiency lift function, which depends on the *reduced frequency*  $k = \frac{\omega U}{b}$ . ' $a$ ' defines the position of the rotation axis with respect to the center of the section and it is dependent on the support condition, the lamination scheme and the applied load. Due to the difficulty in defining its correct value for swept and asymmetric laminated structures, the elastic offset ' $a$ ' is assumed to be null. Theodorsen identified the term ' $C(k)$ ' as a Hankel function of the second kind, which is, in turn, comprised of Bessel functions of the first and second kinds (Rodden, 2012), (Bisplinghoff et al., 1996)

$$C(k) = F(k) + iG(k) = \frac{H_1^{(2)}(k)}{H_1^{(2)}(k) + iH_0^{(2)}(k)} \quad (4.2)$$

A simplified expression of Theodorsen's function was proposed by R.T. Jones in 1940, who presented an exponential approximation of Wagner's indicial solution (1925)

$$C(k) \cong 1 - \frac{0.165}{1 - (0.0455/k)i} - \frac{0.335}{1 - (0.3/k)i} \quad (4.3)$$

The first contribution in Eq.4.1, contains the virtual mass (or noncirculatory) terms that acts as a form of inertia of the fluid surrounding the airfoil and, since the single and doubly-



differentiated terms relative to the mass properties of the structure are small, they are usually neglected. The remaining terms determine the value of the lift and represent the circulatory part. The steady-state simplification can be easily obtained by replacing the function  $C(k)$  with the unitary value and omitting the term containing  $\dot{\alpha}$ .

$$L \cong 2\pi\rho_a U b [\dot{h} + U\alpha] \quad (4.4)$$

For the special purpose of correcting the sectional lift coefficient ' $C_L$ ' to account for the influence of the aspect ratio ('AR= $\frac{2L_w}{c_m}$ ') and the sweep angle (' $\Lambda$ '), Diederich's approximation is used

$$c_{l\alpha} = \frac{dC_L}{d\alpha} = \frac{\pi AR}{\pi AR + c_{l\alpha 0} \cos(\Lambda)} c_{l\alpha 0} \cos(\Lambda) \quad (4.5)$$

where ' $L_w$ ' is the length of the wing, ' $c$ ' is the mean chord (see Fig.5.38) and ' $c_{l\alpha 0}$ ' is the lift-curve slope, which is equal to  $2\pi$ . Finally, since the quantity ' $b\pi$ ' may be approximated with the integral  $\int_{-b}^b \sqrt{\frac{b-x}{b+x}} dx$ , which is able to reproduce the distribution of pressure over a slightly inclined, thin and uncambered airfoil, Eqs.4.1 and 4.4 become, respectively,

$$L = \frac{2\pi AR \cos(\Lambda)}{\pi AR + 2\pi \cos(\Lambda)} \int_{-b}^b \sqrt{\frac{b-x}{b+x}} dx \rho_a U C(k) \left[ \dot{h} + U\alpha + b \left( \frac{1}{2} - a \right) \dot{\alpha} \right] \quad (4.6)$$

$$L \cong \frac{2\pi AR \cos(\Lambda)}{\pi AR + 2\pi \cos(\Lambda)} \int_{-b}^b \sqrt{\frac{b-x}{b+x}} dx \rho_a U [\dot{h} + U\alpha] \quad (4.7)$$

#### 4.1.1 External work for the fixed-wing configuration

The equations of motion of a generic structure oscillating in an incompressible flow can be derived directly using Hamilton's Principle (Eq.3.1). The external work generated by lift can be written as

$$\delta L_{ext} = \int_y \int_x \delta u_z(x, y, z_{top}) L(x, y, z_{top}) dx dy \quad (4.8)$$

where ' $z_{top}$ ' is the upper z-coordinate of the cross-section that, according to Eq.4.1, becomes

$$\delta L_{ext} = cost \left[ \int_y \int_x \delta \mathbf{u}^T \sqrt{\frac{b-x}{b+x}} I_L \dot{\mathbf{u}} + \delta \mathbf{u}^T \sqrt{\frac{b-x}{b+x}} I_L \mathbf{u}_{,x} + \delta \mathbf{u}^T b \left( \frac{1}{2} - a \right) \sqrt{\frac{b-x}{b+x}} I_L \dot{\mathbf{u}}_{,x} \right] dx dy \quad (4.9)$$

where

$$\dot{h} = \dot{u}_z \quad \alpha = \frac{du_z}{dx} = u_{z,x} \quad \dot{\alpha} = \dot{u}_{z,x}$$

and

$$I_L = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad cost = \frac{2\pi AR \cos(\Lambda)}{\pi AR + 2\pi \cos(\Lambda)} \rho_a U C(k)$$

By introducing Eqs.2.9 and 2.15 into Eq.4.9, we obtain

$$\delta L_{ext} = \delta \mathbf{q}_{\tau i}^T \mathbf{D}_L^{ij\tau s} \dot{\mathbf{q}}_{sj} + \delta \mathbf{q}_{\tau i}^T \mathbf{K}_L^{ij\tau s} \mathbf{q}_{sj} + \delta \mathbf{q}_{\tau i}^T \mathbf{D}_{Lm}^{ij\tau s} \dot{\mathbf{q}}_{sj} \quad (4.10)$$

where  $\mathbf{D}_L^{ij\tau s}$ ,  $\mathbf{K}_L^{ij\tau s}$ ,  $\mathbf{D}_{Lm}^{ij\tau s}$  and  $\mathbf{F}_L^{i\tau}$  are the aerodynamic contributions. The first and the third matrices introduce damping into the system, whereas the third matrix represents the aerodynamic stiffness. The last term is a forcing vector that can be overlooked in free-vibration analyses. The aforementioned matrices, written in terms of fundamental nuclei

$$\begin{aligned} \mathbf{D}_L^{ij\tau s} &= cost I_l^{ij} \triangleleft \left( \sqrt{\frac{b-x}{b+x}} F_\tau(x, z_{top}) \mathbf{I}_L F_s(x, z_{top}) \right) \triangleright_x \\ \mathbf{K}_L^{ij\tau s} &= cost I_l^{ij} \triangleleft \left( \sqrt{\frac{b-x}{b+x}} F_{\tau,x}(x, z_{top}) \mathbf{I}_L F_s(x, z_{top}) \right) \triangleright_x \quad \triangleleft \dots \triangleright_x = \int_{-b}^b \dots dx \\ \mathbf{D}_{Lm}^{ij\tau s} &= cost I_l^{ij} b \left( \frac{1}{2} - a \right) \triangleleft \left( \sqrt{\frac{b-x}{b+x}} F_{\tau,x}(x, z_{top}) \mathbf{I}_L F_s(x, z_{top}) \right) \triangleright_x \end{aligned} \quad (4.11)$$

Similarly to rotordynamics equations, the quadratic eigenvalue problem (QEP) of generic order  $R$  is transformed into a classical linear system of  $2 \times R$  order:

$$\begin{cases} \mathbf{M} \ddot{\mathbf{q}} + (\mathbf{D}_L + \mathbf{D}_{Lm}) \dot{\mathbf{q}} + (\mathbf{K} + \mathbf{K}_L) \mathbf{q} = 0 \\ -\dot{\mathbf{q}} + \dot{\mathbf{q}} = 0 \end{cases} \quad (4.12)$$

introducing

$$\mathbf{a} = \begin{Bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{Bmatrix} \quad \dot{\mathbf{a}} = \begin{Bmatrix} \dot{\mathbf{q}} \\ \ddot{\mathbf{q}} \end{Bmatrix} \quad (4.13)$$

the equations of motion assume the form of Eq.3.37 where

$$\mathbf{T}^{-1} \mathbf{R} = \begin{bmatrix} (\mathbf{K} + \mathbf{K}_L)^{-1} (\mathbf{D}_L + \mathbf{D}_{Lm}) & (\mathbf{K} + \mathbf{K}_L)^{-1} \mathbf{M} \\ -\mathbf{I} & \mathbf{0} \end{bmatrix} \quad (4.14)$$

Since the aerodynamic matrices depend on the reduced frequency ' $k$ ', the solutions are found by means of an iterative procedure (Nguyen, 2009). The frequencies are introduced by trial and error for any speed in the considered range, until Eq.3.37 is obtained. The flutter velocity is determined when the real part of one eigenvalue is null.

## 4.2 2-D unsteady aerodynamic theory for rotating lifting surfaces

According to Bielawa (1992), the general expression for the lift distribution on a rotary-wing yields

$$\frac{dL}{dy} = \pi \rho_a b^2 \Omega^2 \left[ \frac{\ddot{h}}{\Omega^2} + \frac{\dot{\alpha}}{\Omega} - b a \frac{\ddot{\alpha}}{\Omega} \right] + 2\pi \rho_a b \Omega^2 C(k) \left[ \frac{\dot{h}}{\Omega} + y \alpha + b \left( \frac{1}{2} - a \right) \frac{\dot{\alpha}}{\Omega} \right] \quad (4.15)$$

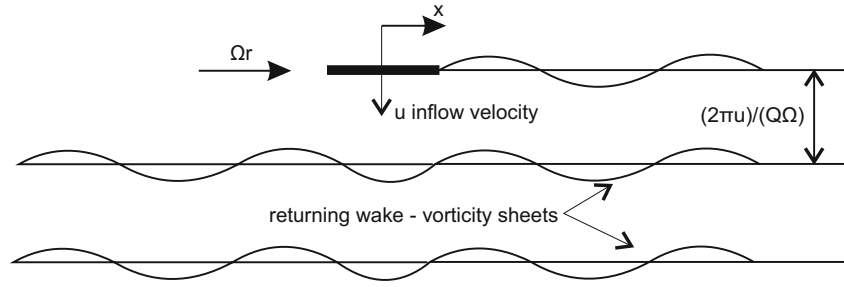
where, in this case, the reduced frequency ' $k$ ' is

$$k = \frac{b}{3R} \frac{\omega}{\Omega}$$

In their study, the authors stated that this approach is limited as it relies on Theodorsen problem. However, it can be easily generalized as the unsteadiness is essentially described by the lift deficiency function. Hence, choosing different expressions of ' $C(k)$ ', it is possible to define several motion conditions. Since the rotating structure aerodynamic is more complex than its fixed-wing counterpart, it is useful to consider the physics of the flowfield near the rotor. The rotor vertical speed (' $V_z$ ') and its thrust ( $\propto C_T$ ) generate an inflow velocity through the disk that can be expressed as

$$u = \Omega R \left[ \frac{V_z}{2} + \sqrt{\frac{V_z^2}{4} \frac{C_T}{2}} \right]$$

In the general case, the rotor blade develops oscillating loads, which generate a radial distribution of vorticity blown downward at velocity ' $u$ '. The wake can be described via models with increasing levels of complexity but, historically, the first successful attempt to incorporate the "returning wake" into the unsteady aerodynamics problem was proposed by Loewy (Loewy, 1957). The aerodynamic actions on a reference blade strongly depend on the shed vorticity of the previous revolutions as shown in Fig.4.1.



**Figure 4.1** Schematic of the Loewy problem (Bielawa, 1992).

According to Loewy's theory, the returning wake is modelled as an infinite number of layers, which are equally spaced by a fixed distance, ' $h_l$ ' semi-chords. Here, the factor ' $h_l$ ' quantify the contribution of the underlying vortices to the lift distribution on the reference blade. The main difference with respect to Theodorsen's model lies in the form of the lift-deficiency function, which is (Dowell et al., 2004):

$$C'(k, m, h_l) = \frac{H_1^{(2)}(k) + 2J_1(k) W(km, h_l)}{H_1^{(2)}(k) + iH_0^{(2)}(k) + 2[J_1(k) + iJ_0(k)]W(km, h_l)} \quad (4.16)$$

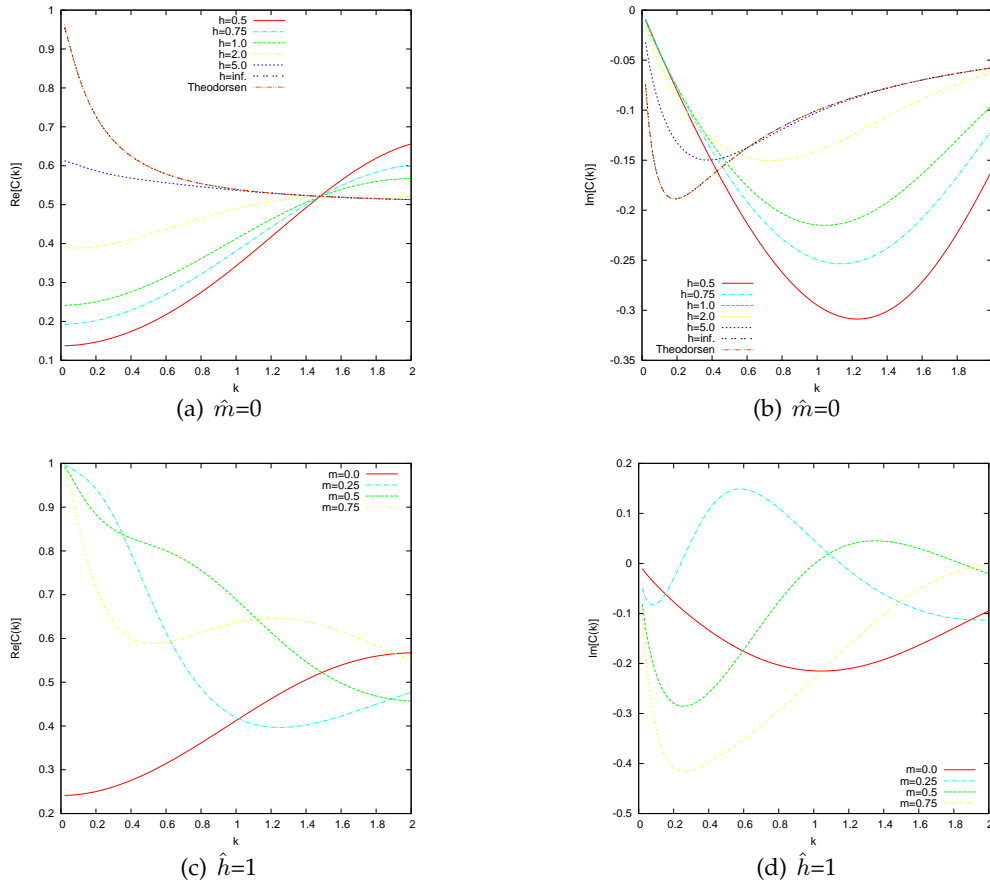
where ' $J_n(k)$ ' and ' $H_n^{(2)}(k)$ ' are the first-kind Bessel and second-kind Hankel functions of order  $n$ , respectively. The wake weighting function ' $W(k, m, h_l)$ ' for a rotor with ' $Q$ ' blades is defined as

$$W(k, m, h) = \frac{1 + \sum_{q=1}^{Q-1} (e^{kh_l Q} e^{i2\pi m})^{Q-q} e^{i\Phi_q}}{e^{kh_l Q} e^{i2\pi m} - 1} \quad (4.17)$$

where ' $m$ ' and ' $\Phi_q$ ' are the ratio of oscillatory frequency to rotational frequency ( $=\omega/\Omega$ ) and the phase angle between the  $q^{th}$  blade and the reference blade. Due to periodicity, Eq.4.17 can be expressed following the formulation for a single-blade rotor with the modified values ' $\hat{m}$ ' and ' $\hat{h}$ ':

$$W(k\hat{m}, \hat{h}) = \frac{1}{e^{k\hat{h}Q} e^{i2\pi\hat{m}} - 1} \quad \hat{m} = \frac{\omega}{Q\Omega} \quad \hat{h} = \frac{2\pi\lambda_0}{Q(b/r)} \quad (4.18)$$

where ' $\lambda_0$ ' is the inflow ratio ( $\simeq \sqrt{C_T/2}$ ) and ' $r$ ' is a generic radial distance. Influence of the frequency ratio and the spacing factor on the Loewy's deficiency function is shown in Fig.4.2. As expected, Loewy's function ' $C(k)$ ' coincide with Theodorsen's expression for high values of spacing factor ' $\hat{h}$ '.



**Figure 4.2** Real and Imaginary parts of the Loewy function vs ' $k$ ' as functions of inflow parameter  $\hat{h}$  (a,b) and frequency ratio  $\hat{m}$  (c,d).

#### 4.2.1 External work for the rotary-wing configuration

The external work defined in Eq.4.9 for a rotating blade is

$$\delta L_{ext} = \text{cost} \left[ \int_y \int_x \delta \mathbf{u}^T \Omega y \sqrt{\frac{b-x}{b+x}} I_L \dot{\mathbf{u}} + \delta \mathbf{u}^T \Omega^2 y^2 \sqrt{\frac{b-x}{b+x}} I_L \mathbf{u}_{,x} dx dy \right] + \text{cost} \left[ \int_y \int_x \delta \mathbf{u}^T \Omega y \sqrt{\frac{b-x}{b+x}} I_L b \left( \frac{1}{2} - a \right) \dot{\mathbf{u}}_{,x} dx dy \right] \quad (4.19)$$

where

$$\mathbf{I}_L = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad cost = \frac{2\pi AR \cos(\Lambda)}{\pi AR + 2\pi \cos(\Lambda)} \rho_a C(k)$$

Introducing, again, the CUF and FEM approximations, we obtain Eq.4.10. The *fundamental nuclei* of the damping and stiffness aerodynamic matrices are

$$\begin{aligned} \mathbf{D}_L^{ij\tau s} &= cost \, \Omega \, I_{l_y}^{ij} \triangleleft \left( \sqrt{\frac{b-x}{b+x}} F_\tau(x, z_{top}) \mathbf{I}_L F_s(x, z_{top}) \right) \triangleright_x \\ \mathbf{K}_L^{ij\tau s} &= cost \, \Omega^2 \, I_{l_y^2}^{ij} \triangleleft \left( \sqrt{\frac{b-x}{b+x}} F_{\tau,x}(x, z_{top}) \mathbf{I}_L F_s(x, z_{top}) \right) \triangleright_x \quad \triangleleft \dots \triangleright_x = \int_{-b}^b \dots dx \\ \mathbf{D}_{Lm}^{ij\tau s} &= cost \, \Omega \, I_{l_y}^{ij} b \left( \frac{1}{2} - a \right) \triangleleft \left( \sqrt{\frac{b-x}{b+x}} F_{\tau,x}(x, z_{top}) \mathbf{I}_L F_s(x, z_{top}) \right) \triangleright_x \end{aligned} \quad (4.20)$$

Velocity varies linearly along the span, therefore yielding the integrals along the  $y$ -direction

$$\left( I_{l_y}^{ij}, I_{l_y^2}^{ij} \right) = \int_l \left( y N_i N_j, y^2 N_i N_j \right) dy$$

A solution is obtained following the procedure presented in Sec.4.2.1.

### 4.3 Panel Flutter of rotating plates: Piston Theory

In this section, the first-order piston theory is applied to rotating structures. Despite its simplicity, the piston theory provides accurate results in the supersonic range, with a Mach number 'M' (= 'flow speed' / 'speed of sound ( $V_s$ )') greater than  $\sqrt{2}$ . The flow on the panel is modelled as a one-dimensional flow in a channel (in a piston) and, no solutions for shock waves downstream of the plate are provided. According to (Carrera and Zappino, 2014a), the local differential pressure 'P' produced by the fluid over the rotating plate surface is

$$P(x, y, z_{top}) = \left( \frac{\rho_a U^2}{\beta} \right) \alpha + \frac{\rho_a U}{\beta} \left( \frac{M^2 - 2}{M^2 - 1} \right) \dot{h} \quad (4.21)$$

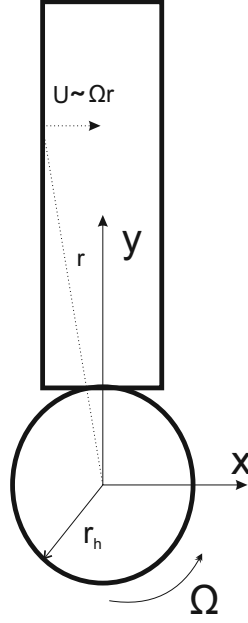
where

$$\begin{aligned} \alpha &= \frac{du_z}{dx} & \dot{h} &= \dot{u}_z \\ M &= \frac{U}{V_s} & U &= \Omega \bar{y} & \beta &= \sqrt{M^2 - 1} \end{aligned}$$

#### 4.3.1 External work for a rotating structure

As previously done, the work due to the external loading is written as

$$\delta L_{ext} = \int_y \int_x \delta u_z(x, y, z_{surf}) P(x, y, z_{surf}) dx dy \quad (4.22)$$



**Figure 4.3** Geometry of the rotating plate.

where  $'z_{surf}'$  is the z-coordinate of the flow-wetted surface. Substituting Eq.4.21 and introducing both CUF and FE approximations, Eq.4.22 becomes

$$\delta L_{ext} = \int_y \int_x \delta \mathbf{u}^T \left( \frac{\rho_a U^2}{\beta} \right) \mathbf{I}_L \mathbf{u}_{,x} + \mathbf{u}^T \left( \frac{\rho_a U}{\beta} \right) \left( \frac{M^2 - 2}{M^2 - 1} \right) \mathbf{I}_L \dot{\mathbf{u}} \, dx \, dy \quad (4.23)$$

From Eq. 4.23, introducing Eq.s 2.9 and 2.15, we obtain

$$\delta L_{ext} = \delta \mathbf{q}_{\tau i}^T \mathbf{K}_L^{ij\tau s} \dot{\mathbf{q}}_{sj} + \delta \mathbf{q}_{\tau i}^T \mathbf{D}_L^{ij\tau s} \dot{\mathbf{q}}_{sj} \quad (4.24)$$

where  $\mathbf{K}_L^{ij\tau s}$  and  $\mathbf{D}_L^{ij\tau s}$  are the aerodynamic damping and stiffness contributions, respectively. Writing these matrices in terms of fundamental nuclei

$$\begin{aligned} \mathbf{K}_L^{ij\tau s} &= I_l^{ij} \triangleleft \left( \left( \frac{\rho_a U^2}{\beta} \right) F_{\tau,x}(x, z_{surf}) \mathbf{I}_L F_s(x, z_{top}) \right) \triangleright_x \\ \mathbf{D}_L^{ij\tau s} &= I_l^{ij} \triangleleft \left( \left( \frac{\rho_a U}{\beta} \right) \left( \frac{M^2 - 2}{M^2 - 1} \right) F_{\tau}(x, z_{surf}) \mathbf{I}_L F_s(x, z_{top}) \right) \triangleright_x \end{aligned} \quad \triangleleft \dots \triangleright_x = \int_{-b}^b \dots \, dx \quad (4.25)$$

Finally, we obtain

$$\mathbf{T}^{-1} \mathbf{R} = \begin{bmatrix} (\mathbf{K} + \mathbf{K}_L + \mathbf{K}_{\Omega} + \mathbf{K}_{\sigma 0})^{-1} (\mathbf{D}_L + \mathbf{G}_{\Omega}) & (\mathbf{K} + \mathbf{K}_L + \mathbf{K}_{\Omega} + \mathbf{K}_{\sigma 0})^{-1} \mathbf{M} \\ -\mathbf{I} & \mathbf{0} \end{bmatrix} \quad (4.26)$$



## Chapter 5

# Numerical Results and discussion

*This chapter aims to present a number of numerical results obtained with the 1D-CUF finite elements. Firstly, static and free-vibration analyses are carried out on structures made of orthotropic and FG materials. Several geometries are studied including compact and thin-walled cross-sections with open and closed profile. Secondly, the results derived by the dynamic analyses of rotating structures are shown highlighting the differences between the classical and refined theories. Finally, the flutter phenomenon for fixed and rotating systems is evaluated for subsonic and supersonic flows.*

### Nomenclature used to denote various expansions

Using the ESL approach, several expansions are evaluated and listed in Table 5.1. In the following, abbreviations stand for

- functions with single trigonometric factor:

$$\begin{aligned} sx &= \sin\left(m\frac{\pi x}{a}\right) & cx &= \cos\left(m\frac{\pi x}{a}\right) & sz &= \sin\left(n\frac{\pi z}{b}\right) & cz &= \cos\left(n\frac{\pi z}{b}\right) \\ sd &= \sin\left(m\frac{\pi xz}{ab}\right) & cd &= \cos\left(m\frac{\pi xz}{ab}\right) \end{aligned}$$

- functions with two trigonometric factors:

$$\begin{aligned} cc &= \cos\left(m\frac{\pi x}{a}\right) \cos\left(n\frac{\pi z}{b}\right) & cs &= \cos\left(m\frac{\pi x}{a}\right) \sin\left(n\frac{\pi z}{b}\right) \\ sc &= \sin\left(m\frac{\pi x}{a}\right) \cos\left(n\frac{\pi z}{b}\right) & ss &= \sin\left(m\frac{\pi x}{a}\right) \sin\left(n\frac{\pi z}{b}\right) \end{aligned}$$

- hyperbolic functions:

$$chx = \cosh(mx) \quad shx = \sinh(mx) \quad chz = \cosh(nz) \quad shz = \sinh(nz)$$

- exponential functions:

$$expx = e^{(mx)} \quad expz = e^{(nz)}$$



◦ polynomial functions:

$$xz = x^m z^n \quad z = z^n$$

The indexes  $m$  and  $n$  indicate summation, and they represent different parameters according to the adopted functions. For trigonometric functions,  $m$  and  $n$  are the half-wave numbers

$$sx = \sin\left(m\frac{\pi x}{a}\right) = \sum_{m=1}^N \sin\left(m\frac{\pi x}{a}\right) = \sin\left(\frac{\pi x}{a}\right) + \sin\left(2\frac{\pi x}{a}\right) + \dots + \sin\left(N\frac{\pi x}{a}\right)$$

$$cc = \cos\left(m\frac{\pi x}{a}\right) \cos\left(n\frac{\pi z}{b}\right) = \sum_{m=0}^N \sum_{n=0}^N \cos\left(m\frac{\pi x}{a}\right) \cos\left(n\frac{\pi z}{b}\right)$$

For exponential and polynomial functions,  $n$  and  $m$  represent exponents:

$$expx = e^{mx} = \sum_{m=1}^N e^{mx} = 1 + e^x + e^{2x} + \dots + e^{Nx}$$

$$xz = \sum_{n=1}^N \sum_{m=1}^{n-1} x^{n-m} z^m = xz + x^2 z + xz^2 + x^3 z + x^2 z^2 + xz^3 + \dots + x^{N-1} z + \dots + xz^{N-1}$$

On the bases of these explicit formulations, the expansions introduced in Tab.5.1 can be written in the extended form

E1-N:

$$\mathbf{u}(x, y, z, t) = \mathbf{u}_1(y, t) + \sin\left(m\frac{\pi x}{a}\right) \mathbf{u}_{(2m)}(y, t) + \sin\left(m\frac{\pi z}{b}\right) \mathbf{u}_{(2m+1)}(y, t)$$

E2-N:

$$\begin{aligned} \mathbf{u}(x, y, z, t) = \mathbf{u}_1(y, t) &+ \sin\left(m\frac{\pi x}{a}\right) \mathbf{u}_{(4m-2)}(y, t) + \sin\left(m\frac{\pi z}{b}\right) \mathbf{u}_{(4m-1)}(y, t) + \\ &+ \cos\left(m\frac{\pi x}{a}\right) \mathbf{u}_{(4m)}(y, t) + \cos\left(m\frac{\pi z}{b}\right) \mathbf{u}_{(4m+1)}(y, t) \end{aligned}$$

E3-N:

$$\begin{aligned} \mathbf{u}(x, y, z, t) = \mathbf{u}_1(y, t) &+ \sin\left(m\frac{\pi x}{a}\right) \mathbf{u}_{(6m-4)}(y, t) + \sin\left(m\frac{\pi z}{b}\right) \mathbf{u}_{(6m-3)}(y, t) + \\ &+ \cos\left(m\frac{\pi x}{a}\right) \mathbf{u}_{(6m-2)}(y, t) + \cos\left(m\frac{\pi z}{b}\right) \mathbf{u}_{(6m-1)}(y, t) + \\ &+ \sin\left(m\frac{\pi xz}{ab}\right) \mathbf{u}_{(6m)}(y, t) + \cos\left(m\frac{\pi xz}{ab}\right) \mathbf{u}_{(6m+1)}(y, t) \end{aligned}$$

E4-N:

$$\mathbf{u}(x, y, z, t) = \mathbf{u}_1(y, t) + e^{(mx)} \mathbf{u}_{(2m)}(y, t) + e^{(mz)} \mathbf{u}_{(2m+1)}(y, t)$$

E5-N:

$$\begin{aligned} \mathbf{u}(x, y, z, t) = \mathbf{u}_1(y, t) &+ e^{(mx)} \mathbf{u}_{(4m)}(y, t) + e^{(mz)} \mathbf{u}_{(4m+1)}(y, t) + \\ &+ \sin\left(m\frac{\pi x}{a}\right) \mathbf{u}_{(4m-2)}(y, t) + \sin\left(m\frac{\pi z}{b}\right) \mathbf{u}_{(4m-1)}(y, t) \end{aligned}$$

E6-N:

$$\mathbf{u}(x, y, z, t) = \mathbf{u}_1(y, t) + \sinh(mx) \mathbf{u}_{(4m-2)}(y, t) + \sinh(mz) \mathbf{u}_{(4m-1)}(y, t) + \\ + \cosh(mx) \mathbf{u}_{(4m)}(y, t) + \cosh(mz) \mathbf{u}_{(4m+1)}(y, t)$$

E7-N:

$$\mathbf{u}(x, y, z, t) = \mathbf{u}_1(y, t) + \sinh(mx) \mathbf{u}_{(6m-4)}(y, t) + \sinh(mz) \mathbf{u}_{(6m-3)}(y, t) + \\ + \cosh(mx) \mathbf{u}_{(6m-2)}(y, t) + \cosh(mz) \mathbf{u}_{(6m-1)}(y, t) + \\ + \sin\left(m\frac{\pi x}{a}\right) \mathbf{u}_{(6m)}(y, t) + \sin\left(m\frac{\pi z}{b}\right) \mathbf{u}_{(6m+1)}(y, t)$$

For the expansion E8 is shown explicitly

E8-3:

$$\mathbf{u}(x, y, z, t) = \mathbf{u}_1(y, t) + \sin\left(1\frac{\pi x}{a}\right) \mathbf{u}_2(y, t) + \sin\left(1\frac{\pi z}{b}\right) \mathbf{u}_3(y, t) + \\ + \sin\left(2\frac{\pi x}{a}\right) \mathbf{u}_4(y, t) + \sin\left(2\frac{\pi z}{b}\right) \mathbf{u}_5(y, t) + xz \mathbf{u}_6(y, t) + \\ + \sin\left(3\frac{\pi x}{a}\right) \mathbf{u}_7(y, t) + \sin\left(3\frac{\pi z}{b}\right) \mathbf{u}_8(y, t) + x^2 z \mathbf{u}_9(y, t) + \\ + xz^2 \mathbf{u}_{10}(y, t)$$

E9-N:

$$\mathbf{u}(x, y, z, t) = \cos\left(m\frac{\pi x}{a}\right) \cos\left(n\frac{\pi z}{b}\right) \mathbf{u}_{m,n}(y, t) + \cos\left(m\frac{\pi x}{a}\right) \sin\left(n\frac{\pi z}{b}\right) \mathbf{u}_{m,n}(y, t) \\ + \sin\left(m\frac{\pi x}{a}\right) \cos\left(n\frac{\pi z}{b}\right) \mathbf{u}_{m,n}(y, t) + \sin\left(m\frac{\pi x}{a}\right) \sin\left(n\frac{\pi z}{b}\right) \mathbf{u}_{m,n}(y, t)$$

E10-N:

$$\mathbf{u}(x, y, z, t) = \mathbf{u}_1(y, t) + z^n \mathbf{u}_{n+1}(y, t)$$

If Taylor-like expansions are added to displacement fields, their order is specified by the subscript. For instance, the first component of  $E2_2$  displacement field becomes

$$u_x = \cos t u_{x_1} + \sin\left(\frac{\pi x}{a}\right) u_{x_2} + \cos\left(\frac{\pi x}{a}\right) u_{x_3} + \sin\left(\frac{\pi z}{b}\right) u_{x_4} + \cos\left(\frac{\pi z}{b}\right) u_{x_5} + \\ + \sin\left(2\frac{\pi x}{a}\right) u_{x_6} + \cos\left(2\frac{\pi x}{a}\right) u_{x_7} + \sin\left(2\frac{\pi z}{b}\right) u_{x_8} + \cos\left(2\frac{\pi z}{b}\right) u_{x_9} + \\ + x u_{x_{10}} + z u_{x_{11}} + x^2 u_{x_{12}} + xz u_{x_{13}} + z^2 u_{x_{14}}$$

## 5.1 Static Analyses

### Convergency Analysis

A cantilever beam with aspect-ratio equal to 100 is here considered. The material is aluminum with Young Modulus and Poisson ratio equal to 73 GPa and 0.34, respectively. The cross-section is square and a concentrated load ( $F_z = -25$  N) is applied at the free tip. The reference solution is provided by the following formula

$$u_z(y = L) = \frac{F_z L^3}{3EI} \quad (5.1)$$

A convergence study is performed to investigate the effect of the number of the mesh elements and the order of the expansion on the transverse displacement evaluated at the loading point (see Tab.5.2). In this case, the convergence is achieved for  $n_e$  equal to 10.

**Table 5.1** Expansions used in the analyses.

|             | <i>E1</i> | <i>E2</i> | <i>E3</i> | <i>E4</i> | <i>E5</i> | <i>E6</i> | <i>E7</i> | <i>E8</i> | <i>E9</i> | <i>E10</i> |
|-------------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|------------|
| <i>cost</i> | ✓         | ✓         | ✓         | ✓         | ✓         | ✓         | ✓         | ✓         | ✓         | ✓          |
| <i>sx</i>   | ✓         | ✓         | ✓         |           | ✓         |           | ✓         | ✓         |           |            |
| <i>cx</i>   |           | ✓         | ✓         |           |           |           |           |           |           |            |
| <i>sz</i>   | ✓         | ✓         | ✓         |           | ✓         |           | ✓         | ✓         |           |            |
| <i>cz</i>   |           | ✓         | ✓         |           |           |           |           |           |           |            |
| <i>sd</i>   |           |           | ✓         |           |           |           |           |           |           |            |
| <i>cd</i>   |           |           | ✓         |           |           |           |           |           |           |            |
| <i>expx</i> |           |           |           | ✓         | ✓         |           |           |           |           |            |
| <i>expz</i> |           |           |           | ✓         | ✓         |           |           |           |           |            |
| <i>shx</i>  |           |           |           |           |           | ✓         | ✓         |           |           |            |
| <i>chx</i>  |           |           |           |           |           | ✓         | ✓         |           |           |            |
| <i>shz</i>  |           |           |           |           |           | ✓         | ✓         |           |           |            |
| <i>chz</i>  |           |           |           |           |           | ✓         | ✓         |           |           |            |
| <i>cc</i>   |           |           |           |           |           |           |           |           | ✓         |            |
| <i>cs</i>   |           |           |           |           |           |           |           |           | ✓         |            |
| <i>sc</i>   |           |           |           |           |           |           |           |           | ✓         |            |
| <i>ss</i>   |           |           |           |           |           |           |           |           | ✓         |            |
| <i>xz</i>   |           |           |           |           |           |           |           | ✓         |           |            |
| <i>z</i>    |           |           |           |           |           |           |           |           |           | ✓          |

**Table 5.2** Effect of the number ( $n_e$ ) of elements B4 on  $-u_z (\times 10^{-3} [m])$ . Loading case: bending.

| N                     | $n_e = 1$ | $n_e = 5$ | $n_e = 10$ | $n_e = 30$ |
|-----------------------|-----------|-----------|------------|------------|
| <i>CBT</i>            | 1.3703    | 1.3703    | 1.3703     | 1.3703     |
| <i>FOBT</i>           | 1.3704    | 1.3704    | 1.3704     | 1.3704     |
| <i>TE<sub>1</sub></i> | 1.3704    | 1.3704    | 1.3704     | 1.3704     |
| <i>TE<sub>3</sub></i> | 1.2728    | 1.3442    | 1.3534     | 1.3597     |
| <i>TE<sub>5</sub></i> | 1.2840    | 1.3541    | 1.3627     | 1.3684     |
| Ref.eq.(5.1)=1.3698   |           |           |            |            |

*CBT* stands for Classical beam theory.

*FOBT* stands for First-order beam theory.

### 5.1.1 Laminated compact beams

This section shows the results obtained for cantilevered symmetric  $(0^0/90^0/0^0)$  and antisymmetric  $(0^0/90^0)$  cross-ply laminated beams constituted by orthotropic material. All laminae have the same thickness and, if not otherwise specified, the dimensionless material properties adopted are:

$$E_A/E_T = 25 \quad G_{AT}/G_{TT} = 2.5 \quad \nu_{AT} = 0.1 \quad \nu_{TT} = 0.3$$

where  $A$  refers to the fiber direction and  $T$  refers to the normal direction.

For sake of convenience, the results are presented in non-dimensional form:

$$\bar{u}_z = 100 \frac{bh^3 E_T}{q_0 L^4} u_z, \quad \bar{\sigma}_{ij} = \frac{\sigma_{ij}}{q_0}, \quad \text{with} \quad i, j = x, y, z$$

where  $L$  is the length of the beam,  $b$  and  $h$  the dimensions of the rectangular cross-section and  $q_0$  the intensity of the distributed load. The structures are modelled with ten finite elements. The displacement of the tip, the stresses  $\bar{\sigma}_{yy}$  and  $\bar{\sigma}_{zz}$  are compared with those obtained by using the solid elements *HEX20* of *MSC NASTRAN*® (Tab.5.3 and 5.4). For both structures, it is worth noting that expansion  $E4$ , where exponential functions are used, leads to results that strongly agree with the reference solutions. With a comparable number of degrees of freedom, the expansion  $E1$  provides good results both in term of displacement and longitudinal stress  $\bar{\sigma}_{yy}$  whereas, for stress  $\bar{\sigma}_{zz}$ , the solution appears inadequate. Furthermore, in the last column of Tab.5.3, the ratio between the CPU time using the expansion and Euler-Bernoulli theory is shown. These values give information about the real computational effort required by refined theories. Although the ratios can reach high values (200-300), the computation time with a common laptop is around few seconds.

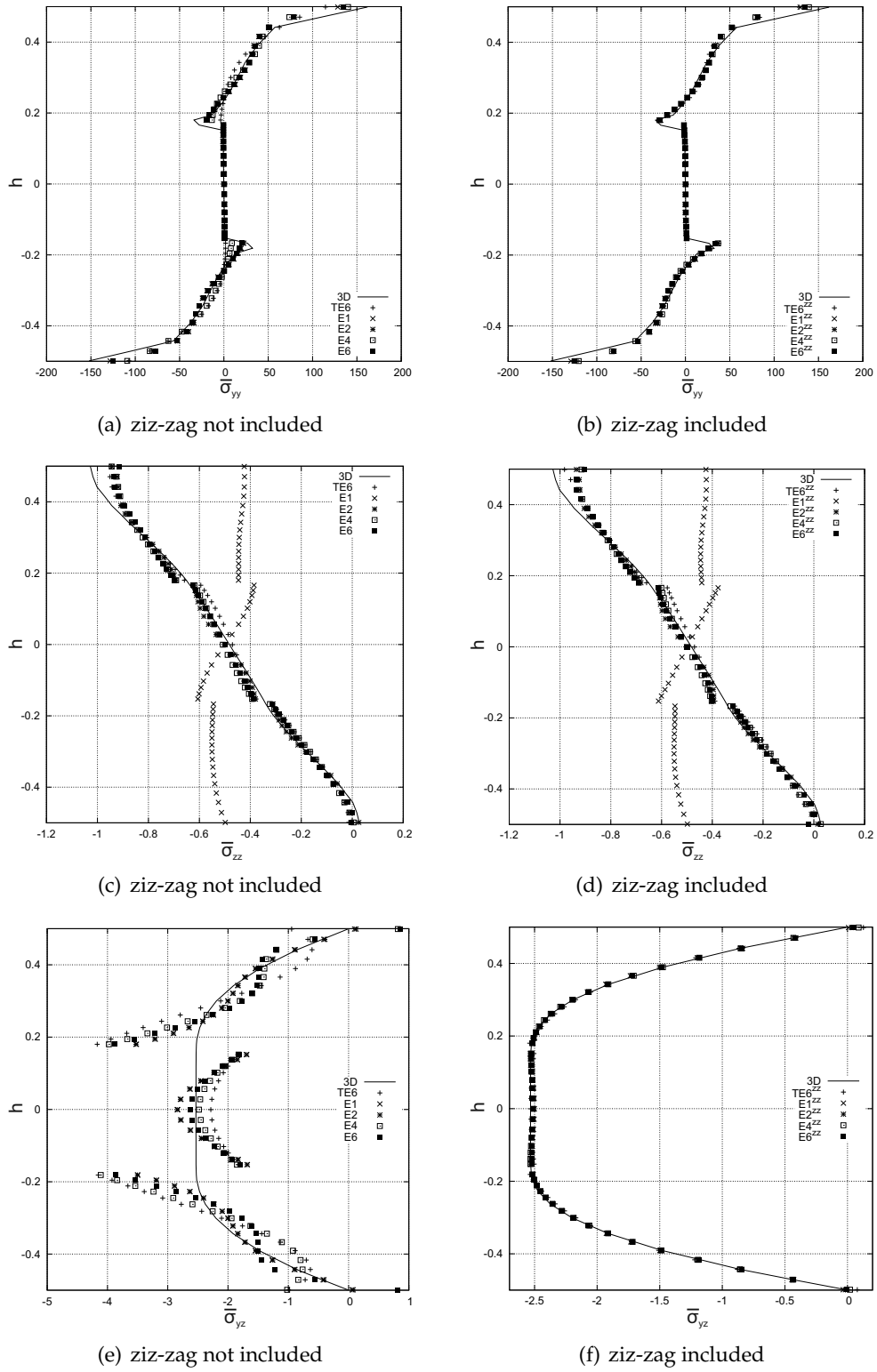
The importance of the zig-zag function is evident in Figs.5.1 and 5.2 where distributions through-the-thickness of the non-dimensional stresses ( $\bar{\sigma}_{yy}$ ,  $\bar{\sigma}_{zz}$  and  $\bar{\sigma}_{yz}$ ) are shown for several kinds of expansions. Figures labeled with  $(a-c-e)$  refer to expansions without the zig-zag term whereas figures  $(b-d-f)$  show the distributions when this term is included. Results demonstrate that the combination of different functions by introducing the discontinuity of the first derivative of displacements allows to obtain quasi-3D solutions with very low computational efforts.

**Table 5.3** Non-dimensional displacements and stresses of a symmetric cross-ply beam under an uniform distributed load.  $\sigma_{yy}$  at  $(b/2; 0; -0.367h)$  and  $\sigma_{zz}$  at  $(b/2; L/2; h/2)$ .  $L/h = 4$

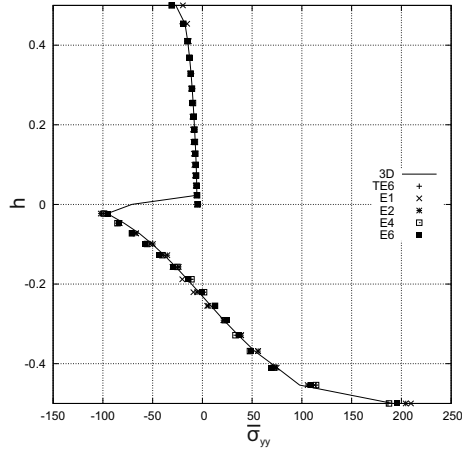
|                | $N$      | $-\bar{u}_z$ | $-\bar{\sigma}_{yy}^-$ | $-\bar{\sigma}_{zz}^+$ | $DOFs$ | $t/t_{CBT}$ |
|----------------|----------|--------------|------------------------|------------------------|--------|-------------|
| <i>Nastran</i> |          | 17.9754      | 30.5844                | 1.0278                 | 103920 |             |
| <i>EBBT</i>    |          | 6.2245       | 36.5184                | 0.0000                 | 279    | 1           |
| <i>FSDT</i>    |          | 14.0223      | 36.5184                | 0.0000                 | 279    | 1           |
| <i>TE1</i>     |          | 14.0224      | 36.5184                | 0.4958                 | 279    | 4           |
|                | $zz$     | 17.3328      | 38.0646                | 0.4475                 | 372    | 6           |
| <i>TE2</i>     |          | 14.0154      | 36.3298                | 0.9368                 | 558    | 13          |
|                | $zz$     | 17.3060      | 37.8849                | 1.0556                 | 651    | 16          |
| <i>TE3</i>     |          | 16.7647      | 28.3861                | 1.0439                 | 930    | 33          |
|                | $zz$     | 17.7591      | 31.1710                | 1.0203                 | 1023   | 40          |
| <i>TE6</i>     |          | 17.1384      | 25.2949                | 0.9950                 | 2604   | 271         |
|                | $zz$     | 17.8371      | 27.3629                | 1.0323                 | 2697   | 273         |
| <i>E1</i>      | 5        | 17.5913      | 31.1008                | 0.4238                 | 1023   | 55          |
|                | $5^{zz}$ | 17.9017      | 30.0005                | 0.4251                 | 1116   | 63          |
| <i>E2</i>      | 5        | 17.5352      | 30.9342                | 0.9435                 | 1953   | 148         |
|                | $5^{zz}$ | 17.8420      | 29.8032                | 0.9352                 | 2046   | 161         |
| <i>E4</i>      | 7        | 17.3130      | 26.6598                | 0.9450                 | 1395   | 76          |
|                | $7^{zz}$ | 17.8397      | 26.2546                | 0.9135                 | 1488   | 86          |
| <i>E6</i>      | 4        | 17.4534      | 32.3089                | 0.9145                 | 1581   | 100         |
|                | $4^{zz}$ | 17.8407      | 29.2543                | 0.9013                 | 1674   | 122         |

**Table 5.4** Non-dimensional displacements and stresses of an antisymmetric cross-ply beam under an uniform distributed load.  $\sigma_{yy}$  at  $(b/2;0;-0.328h)$  and  $\sigma_{zz}$  at  $(b/2;L/2;h/2)$ .  $L/h = 4$

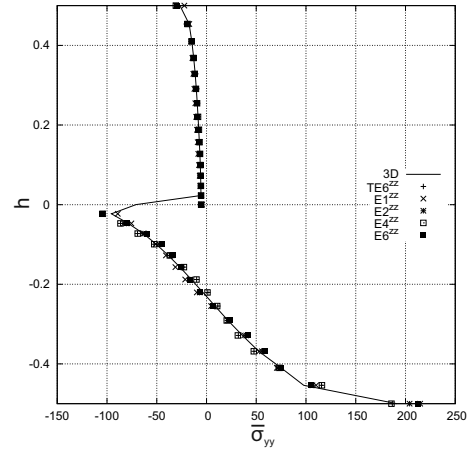
|                | $N$      | $\bar{u}_z$ | $\bar{\sigma}_{yy}^-$ | $-\bar{\sigma}_{yy}^+$ | $-\bar{\sigma}_{zz}^+$ | $DOFs$ |
|----------------|----------|-------------|-----------------------|------------------------|------------------------|--------|
| <i>Nastran</i> |          | 43.0937     | 36.1128               | 11.4270                | 1.0239                 | 198300 |
| <i>EBBT</i>    |          | 31.9645     | 50.0290               | 11.4394                | 0.0000                 | 279    |
| <i>FSDT</i>    |          | 40.8757     | 50.0290               | 11.4394                | 0.0000                 | 279    |
| <i>TE1</i>     |          | 40.8609     | 50.0290               | 11.4394                | 0.3887                 | 279    |
|                | $zz$     | 41.2077     | 48.2077               | 13.0898                | 0.6001                 | 372    |
| <i>TE2</i>     |          | 40.9202     | 50.3749               | 13.2647                | 0.9811                 | 558    |
|                | $zz$     | 41.0123     | 48.3814               | 13.6445                | 0.9270                 | 651    |
| <i>TE3</i>     |          | 41.6311     | 36.7505               | 12.8075                | 1.1380                 | 930    |
|                | $zz$     | 41.6610     | 39.2040               | 11.2977                | 1.0460                 | 1023   |
| <i>TE6</i>     |          | 42.1657     | 33.9530               | 10.8486                | 0.9599                 | 2604   |
|                | $zz$     | 42.2660     | 32.0183               | 10.7739                | 0.9232                 | 2697   |
| <i>E1</i>      | 5        | 41.0128     | 38.9804               | 12.0684                | 0.1919                 | 1023   |
|                | $5^{zz}$ | 41.9098     | 37.8884               | 13.2106                | 0.4934                 | 1116   |
| <i>E2</i>      | 5        | 42.2677     | 38.8214               | 11.9479                | 0.9503                 | 1953   |
|                | $5^{zz}$ | 42.3232     | 39.9750               | 11.8412                | 0.9267                 | 2046   |
| <i>E4</i>      | 7        | 42.2131     | 33.2601               | 11.7440                | 0.9671                 | 1395   |
|                | $7^{zz}$ | 42.2895     | 31.5211               | 11.8038                | 0.9559                 | 1488   |
| <i>E6</i>      | 4        | 42.2211     | 36.4385               | 11.4997                | 0.8818                 | 1581   |
|                | $4^{zz}$ | 42.3144     | 41.9019               | 11.1968                | 0.9304                 | 1674   |



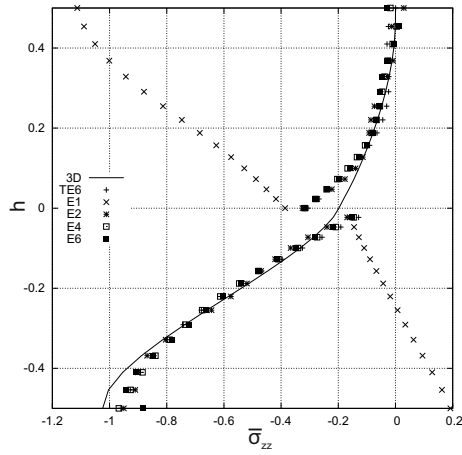
**Figure 5.1** Distribution of stresses  $\bar{\sigma}$  through-the-thickness of a symmetric cantilever beam ( $L/h=4$ ).



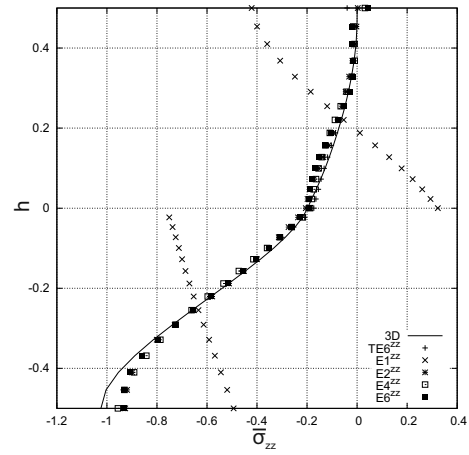
(a) ziz-zag not included



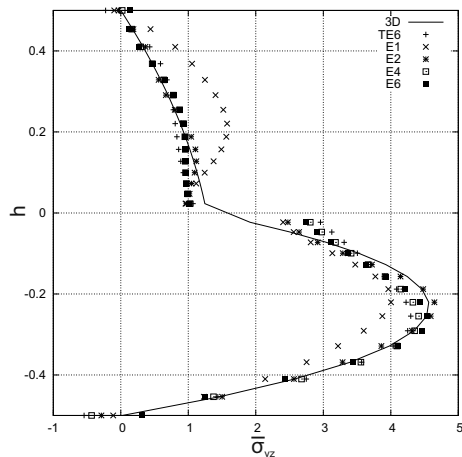
(b) ziz-zag included



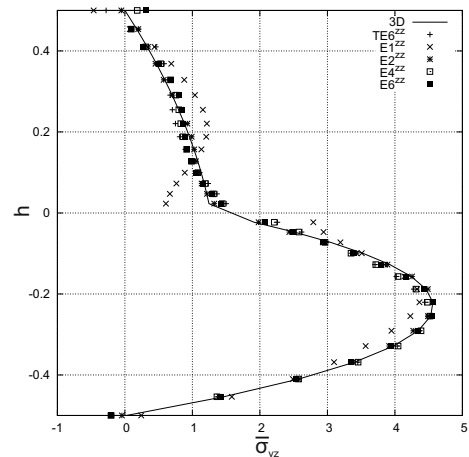
(c) ziz-zag not included



(d) ziz-zag included



(e) ziz-zag not included



(f) ziz-zag included

**Figure 5.2** Distribution of stresses  $\bar{\sigma}$  through-the-thickness of an anti-symmetric cantilever beam ( $L/h=4$ ).



## 5.1.2 Functionally Graded Structures

### The rectangular cross-section FG beam

First of all, a rectangular cross-section beam is considered to assess the theory. The aspect ratio of the cross-section ( $a/h$ ) is assumed to be equal to 0.01 in order to neglect the effects in the width-wise direction. The structure is subjected to a sinusoidal pressure that can act either on the top ( $z_T$ ) or bottom ( $z_B$ ) surface, whose distribution is

$$p(x, y) = p_{zz} \sin\left(\frac{\pi}{L} y\right) \quad p_{zz} = 1 \quad (5.2)$$

where 'L' is the length of the beam. The Young's modulus  $E(x, z)$  varies with the exponential law in Eq.5.3 so that the ratio  $E(x, z_T)/E(x, z_B)=10$  with  $E(x, z_B) = 1000$ .

$$E(x, z) = E(x, z_B) e^{\alpha_1 x + \beta_1} e^{\alpha_2 z + \beta_2} \quad (5.3)$$

Two length-to-thickness ratios ( $L/h$ ) are considered and the results are compared with those available in Giunta et al. (2010). In their analysis, Giunta *et al.* provided closed form solutions using CUF, and higher-order displacement theories were obtained adopting Taylor-type expansions: the relative results were validated by solutions available in the literature. The transverse displacement and stresses are reported in the following non-dimensional forms

$$\bar{u}_z = \frac{u_z}{h} \quad \bar{\sigma}_{ij} = \frac{\sigma_{ij}}{p_{zz}}$$

The structure is simply supported and it is modelled using ten 4-node beam (B4) elements. Tables 5.5 and 5.6 show the results for  $L/h$  equal to 10 and 2.5, respectively.

**Table 5.5** Transverse displacement and stresses for  $L/h = 10$ ,  $\nu=0.25$  and loading acting at  $z_T$

| Theory                         | $10\bar{u}_z(L/2, z_B)$ | $\bar{\sigma}_{yy}(L/2, -h/2)$ | $\bar{\sigma}_{yz}(0, 0)$ | DOF  |
|--------------------------------|-------------------------|--------------------------------|---------------------------|------|
| EBBT Ref.(Giunta et al., 2010) | -4.0311                 | 26.928                         | -                         |      |
| FSDT Ref.(Giunta et al., 2010) | -4.0959                 | 26.928                         | -2.5753                   |      |
| TE3 Ref.(Giunta et al., 2010)  | -4.0938                 | 26.942                         | -4.3468                   |      |
| TE7 Ref.(Giunta et al., 2010)  | -4.0942                 | 26.922                         | -4.4527                   |      |
| EBBT                           | -4.0308                 | 26.926                         | -                         | 279  |
| FSDT                           | -4.0955                 | 26.926                         | -2.5816                   | 279  |
| TE3                            | -4.0934                 | 26.940                         | -4.3530                   | 930  |
| TE7                            | -4.0938                 | 26.920                         | -4.4588                   | 3348 |
| E2-5 <sub>2</sub>              | -4.0938                 | 26.922                         | -4.4587                   | 2418 |
| E10-4 <sub>2</sub>             | -4.0938                 | 26.923                         | -4.4583                   | 930  |
| E10-9 <sub>2</sub>             | -4.0938                 | 26.922                         | -4.4587                   | 1395 |

EBBT: Euler-Bernoulli Beam Theory.

FSDT: First-order Shear Deformation Theory.

-: results not provided by the model.

As far as classical and TE beam models are concerned, the finite element solutions agree with the analytical approach for both cases. Moreover, Fig. 5.3 shows the stress distributions through the thickness of the shortest beam for different displacement models.

**Table 5.6** Transverse displacement and stresses for  $L/h = 2.5$ ,  $\nu=0.3$  and loading acting at  $z_B$ 

| Theory                         | $\sigma_{yy}^-(L/2, -h/2)$ | $\sigma_{yy}^-(L/2, h/2)$ | $\sigma_{yz}^{max}$ | DOF  |
|--------------------------------|----------------------------|---------------------------|---------------------|------|
| EBBT Ref.(Giunta et al., 2010) | -1.6830                    | 8.0363                    | -                   |      |
| FSDT Ref.(Giunta et al., 2010) | -1.6830                    | 8.0360                    | 2.0359              |      |
| TE3 Ref.(Giunta et al., 2010)  | -1.8354                    | 8.5441                    | 1.3554              |      |
| TE8 Ref.(Giunta et al., 2010)  | -1.8175                    | 8.4440                    | 1.2822              |      |
| EBBT                           | -1.6830                    | 8.0363                    | -                   | 279  |
| FSDT                           | -1.6830                    | 8.0363                    | 2.0362              | 279  |
| TE3                            | -1.8355                    | 8.5445                    | 1.3548              | 930  |
| TE8                            | -1.8175                    | 8.4420                    | 1.2807              | 4185 |
| E2-5 <sub>2</sub>              | -1.8615                    | 8.3784                    | 1.2750              | 2418 |
| E10-4 <sub>2</sub>             | -1.8620                    | 8.3809                    | 1.2753              | 930  |
| E10-9 <sub>2</sub>             | -1.8629                    | 8.3827                    | 1.2750              | 1395 |

*EBBT*: Euler-Bernoulli Beam Theory.

*FSDT*: First-order Shear Deformation Theory.

-: results not provided by the model.

It should be noted that the E10-4<sub>2</sub> expansion leads to accurate results with a very low number of degrees of freedom (DOF<sup>1</sup>). This is due to the fact that the beam can be considered in a state of plane stress (in  $yz$ -plane), which allows to disregard the  $x$ -terms in the TE expansions since the transverse stresses are dominant over those along the width.

### The multi-layer FG beam

A cantilevered three-layer beam is here considered. The length ' $L$ ', the width ' $a$ ' and the whole thickness ' $h$ ' of the structure are 2.5, 0.5 and 1 m, respectively. The layers are made of functionally graded materials, whose Young's moduli ' $E(x,z)$ ' follow three different gradation functions (linear, exponential and sigmoid):

$$-0.5 < z < -0.3 : E(z) = 400 \times 10^9 z + 270 \times 10^9 ;$$

$$-0.3 < z < 0.0 : E(z) = 150 \times 10^9 e^{\frac{z}{0.3} \left( \ln\left(\frac{250}{150}\right) + \frac{250}{150} \right)} ;$$

$$0.0 < z < 0.25 : E(z) = 250 \times 10^9 V(z) + 380 \times 10^9 (1 - V(z)) \text{ where } V(z) = 1 - \frac{1}{2} \left( 1 - \frac{2(z-0.25)}{0.5} \right)^3$$

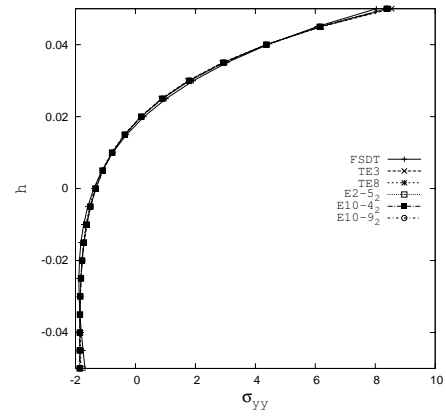
$$0.25 < z < 0.5 : E(z) = 250 \times 10^9 V(z) + 380 \times 10^9 (1 - V(z)) \text{ where } V(z) = \frac{1}{2} \left( 1 + \frac{2(z-0.25)}{0.5} \right)^3$$

The Poisson's ratio is considered constant and equal to 0.25 for all layers and the structure undergoes to a distributed pressure equal to 1 MPa. The beam is modelled with ten 4-node finite elements along the  $y$ -axis. For comparison purposes, a finite element solution is provided, where solid elements HEX 8 are used. The mechanical properties are defined in the centroid of each element and, in order to correctly describe the materials distribution, 25, 10 and 50

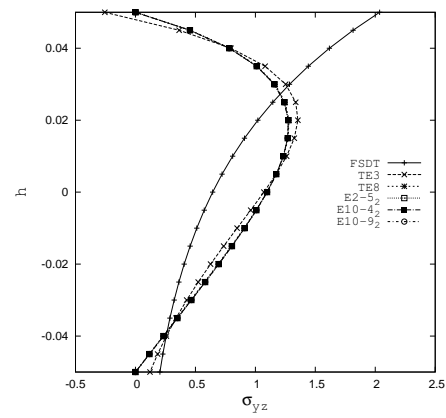
<sup>1</sup> DOF =  $3 \times N_{nodes} \times T$ , in which ' $N_{nodes}$ ' is the number of structural nodes and ' $T$ ' the number of terms in the expansion.

elements are used along the length, the width and the thickness, respectively. The total number of degrees of freedom of the 3D model is therefore equal to 42075. Figures 5.4 show the transverse and axial displacements obtained with a number of beam theories in which polynomial, trigonometric as well as exponential functions are used. Because of shear effects, the transverse displacement has a non-linear variation along the thickness, and the beam results are slightly lower than the solid solution. Moreover, it is noteworthy that the FSDT predicts a constant value of  $u_z$ . The axial displacements in Fig. 5.4-b perfectly match the reference solution. Regarding the stress distributions, Fig.5.5 shows that the considered refined displacement theories lead to quite accurate results. However, slightly different stress distributions are presented in Fig.5.6.

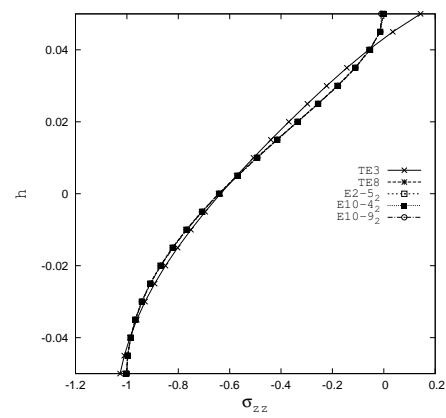
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(a)

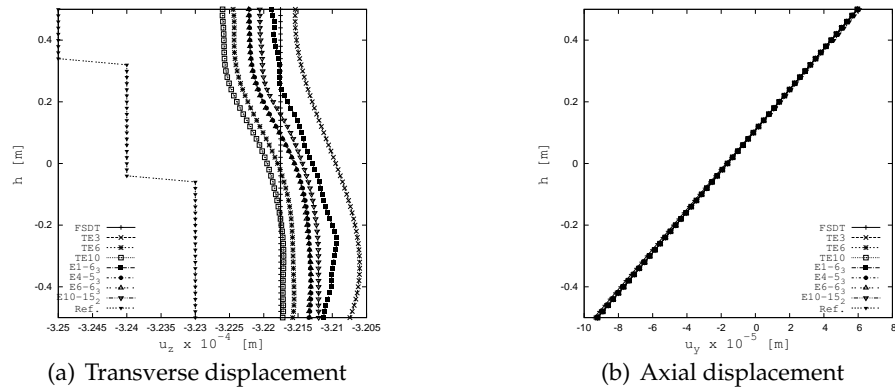


(b)

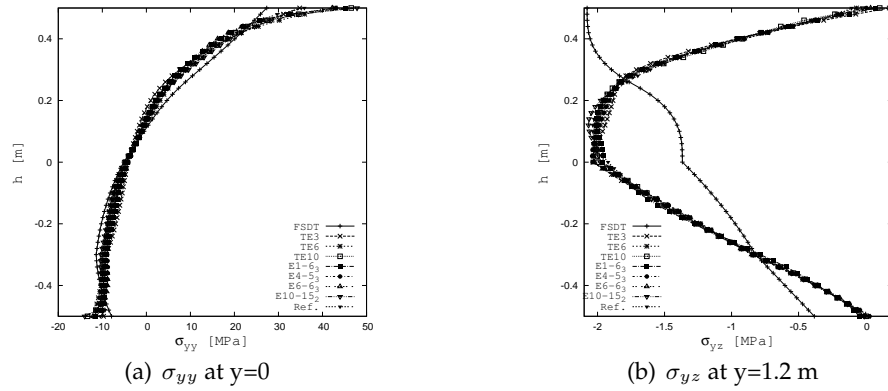


(c)

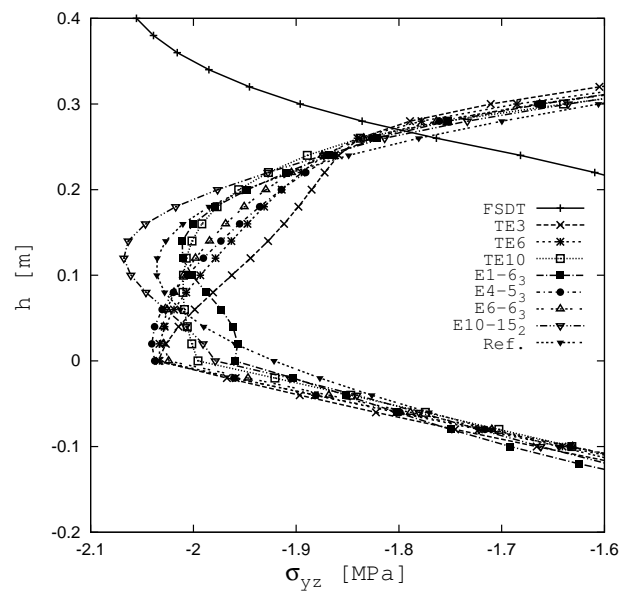
**Figure 5.3** Stress distributions through the thickness. FGM beam aspect-ratio 2.5.



**Figure 5.4** Displacements along the thickness of the three-layered FG beam at  $y=L$ .



**Figure 5.5** Stresses along the thickness of the three-layered FG beam.



**Figure 5.6** Zoom of the shear stress  $\sigma_{yz}$  distribution along the thickness of the three-layered FG beam.

## 5.2 Free-Vibration Analyses

### 5.2.1 Laminated compact beams

In the first two examples, symmetric ( $0^0/90^0/0^0$ ) and antisymmetric ( $0^0/90^0$ ) cross-ply laminated beams are studied. The effect of the beam dimensions and lamination on the natural frequencies and modes as well as on the accuracy of the proposed theories is investigated. The simply supported beams are constituted by orthotropic material, whose properties are:

$$E_L/E_T = 25 \quad G_{LT}/G_{TT} = 2.5 \quad \nu_{LT} = \nu_{TT} = 0.25$$

where  $L$  refers to the fiber direction and  $T$  refers to the normal direction. All laminae are assumed to be of the same thickness. For convenience, the natural angular frequencies<sup>2</sup> are presented in dimensionless form:

$$\bar{\omega} = \frac{L^2}{b} \sqrt{\frac{\rho}{E_T}} \omega$$

where  $L$  is the length of the beam and  $b$  the width of cross-section. The results are compared with those found in (Giunta et al., 2013) where the authors provided closed form solutions using CUF. The higher-order displacement theories were obtained adopting Taylor-type expansions and the relative results were assessed with three-dimensional FEM solutions. In Tabs.5.7 and 5.8, the dimensionless frequencies of the symmetric beams are shown, for length-to-thickness ratios equal to 10 and 5, respectively. The first two modal shapes are pure flexural modes, which occur in different planes. When the beam becomes thicker, these modes interchange their order of appearance and, only with higher order displacement models, the "modal swapping" becomes detectable. For both cases, the use of zig-zag function substantially improves the accuracy of results for the flexural modal shape in  $yz$ -plane, while, for the remaining mode of deformation, the most efficient models are the Taylor-type expansions and E8. By contrast, the third mode involves a torsional deformation, whose natural frequency is computed with acceptable accuracy adopting the theories with the  $xz$  term (in other words, Taylor-type expansions and E8). The last two modes involves shear deformations which are typically detected by using 3D elements. For the sake of clarity, Fig.5.7 shows these modal shapes computed with the expansion TE6. The axial/shear deformation appears as fourth and fifth mode, respectively. By comparing the results it is shown that the frequency values obtained with TE8 and E2 are close to the reference solutions for both values of length-to thickness ratios. Regarding the last shear modes, the zig-zag function once again enables a significant improvement of the solutions, indeed, if we consider TE3 expansion for  $L/h$  equal to 10, the relative error decreases from 6% to 0.2%. An antisymmetric laminated beam with length-to-thickness ratio equal to 5 is considered. The dimensionless angular frequencies are shown in Tab.5.9. For this case, the five modal shapes are: flexural on  $yz$ -plane, flexural/torsional on  $xy$ -plane, torsional, axial/shear and shear on  $xy$ -plane, respectively. For sake of clarity, the latter two deformation modes are shown in Fig.5.7-c and 5.7-d. Classical models are intrinsically unable to detect the torsional mode and the shear mode, hence, variable kinematic models are needed. As in the previous examples, the accuracy of the computation of frequencies is affected by the choice of the functions as well as by the kind of modal shapes. For instance, TE6 and E8 yield precise results for torsional and coupled modes (modes II and III), E2, E3 and E6 seem suitable for

<sup>2</sup> The angular frequency is  $\omega = 2\pi f$  (rad/sec) whereas  $f$  is the frequency (Hz).

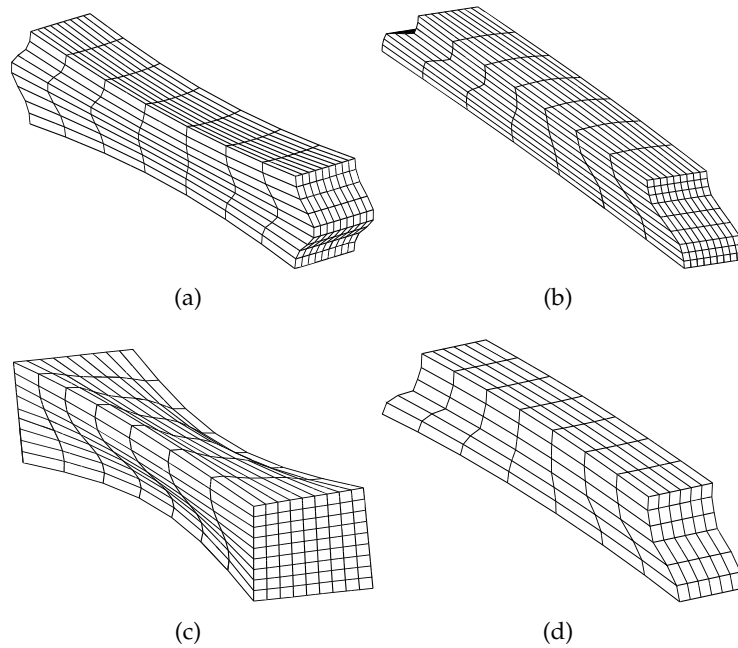
detecting the axial/shear mode (IV), whereas the zig-zag function is more effective for shear modes (modes IV and V).

**Table 5.7** Dimensionless natural frequencies,  $L/b=10$ ,  $[0/90/0]$  beam.

|  | Mode I | Mode II | Mode III | Mode IV | Mode V | DOFs |
|--|--------|---------|----------|---------|--------|------|
| TE <sub>23</sub> (Giunta et al., 2013) | 10.038 | 10.371  | 17.719   | 126.70  | 219.02 | 900  |
| EBBT                                   | 11.697 | 13.929  | -        | 129.52  | -      | 198  |
| FSDT                                   | 10.358 | 11.418  | -        | 129.58  | 268.47 | 198  |
| TE2                                    | 10.367 | 11.420  | 20.968   | 128.46  | 268.50 | 396  |
| TE2 <sup>zz</sup>                      | 10.367 | 10.450  | 20.968   | 128.75  | 221.10 | 462  |
| TE3                                    | 10.113 | 10.637  | 20.968   | 128.78  | 233.05 | 660  |
| TE3 <sup>zz</sup>                      | 10.113 | 10.337  | 20.968   | 128.74  | 218.85 | 726  |
| TE6                                    | 10.069 | 10.501  | 18.005   | 127.24  | 224.88 | 1848 |
| TE6 <sup>zz</sup>                      | 10.069 | 10.333  | 18.005   | 127.22  | 218.67 | 1914 |
| E1-5                                   | 10.176 | 10.426  | 21.075   | 129.61  | 220.32 | 726  |
| E1-5 <sup>zz</sup>                     | 10.176 | 10.351  | 21.075   | 129.61  | 218.91 | 792  |
| E2-4                                   | 10.174 | 10.416  | 21.088   | 127.86  | 220.24 | 1122 |
| E2-4 <sup>zz</sup>                     | 10.174 | 10.342  | 21.088   | 126.91  | 218.66 | 1188 |
| E3-3                                   | 10.173 | 10.416  | 21.032   | 128.18  | 220.23 | 1254 |
| E3-3 <sup>zz</sup>                     | 10.173 | 10.340  | 21.031   | 127.20  | 218.66 | 1320 |
| E5-2                                   | 10.188 | 10.491  | 21.475   | 128.07  | 258.00 | 594  |
| E5-2 <sup>zz</sup>                     | 10.188 | 10.350  | 21.471   | 128.61  | 218.95 | 660  |
| E6-2                                   | 10.173 | 10.656  | 21.075   | 127.99  | 233.73 | 594  |
| E6-2 <sup>zz</sup>                     | 10.173 | 10.342  | 21.075   | 125.78  | 218.94 | 660  |
| E7-2                                   | 10.173 | 10.428  | 21.075   | 126.18  | 258.00 | 858  |
| E8-5                                   | 10.073 | 10.417  | 18.242   | 129.60  | 220.10 | 1386 |
| E8-5 <sup>zz</sup>                     | 10.073 | 10.342  | 18.241   | 129.22  | 218.68 | 1452 |

**Table 5.8** Dimensionless natural frequencies,  $L/b=5$ ,  $[0/90/0]$  beam.

|                                 | Mode I | Mode II | Mode III | Mode IV | Mode V |
|---------------------------------|--------|---------|----------|---------|--------|
| $TE_{23}$ (Giunta et al., 2013) | 6.9252 | 7.5017  | 9.0683   | 55.914  | 58.135 |
| EBBT                            | 13.752 | 11.552  | -        | -       | 64.722 |
| FSDT                            | 8.0853 | 8.0409  | -        | -       | 64.752 |
| TE2                             | 8.0837 | 8.0451  | 10.502   | 67.238  | 62.857 |
| $TE2^{zz}$                      | 7.0131 | 8.0451  | 10.502   | 66.319  | 62.893 |
| TE3                             | 7.1597 | 7.6230  | 10.502   | 59.912  | 62.824 |
| $TE3^{zz}$                      | 6.8990 | 7.6230  | 10.502   | 56.467  | 62.540 |
| TE8                             | 6.9800 | 7.5279  | 9.1129   | 56.694  | 58.670 |
| $TE8^{zz}$                      | 6.8886 | 7.5279  | 9.1129   | 55.729  | 58.886 |
| E1-5                            | 6.9675 | 7.6956  | 10.536   | 58.245  | 64.591 |
| $E1-5^{zz}$                     | 6.8943 | 7.6956  | 10.535   | 57.322  | 64.552 |
| E2-4                            | 6.9653 | 7.6952  | 10.537   | 58.389  | 59.600 |
| $E2-4^{zz}$                     | 6.8913 | 7.6952  | 10.537   | 57.323  | 58.670 |
| E5-2                            | 7.0480 | 7.7013  | 10.586   | 58.608  | 62.850 |
| $E5-2^{zz}$                     | 6.8969 | 7.7013  | 10.585   | 57.361  | 62.621 |
| E6-2                            | 7.1752 | 7.6954  | 10.536   | 60.916  | 64.457 |
| $E6-2^{zz}$                     | 6.8973 | 7.6954  | 10.535   | 57.425  | 64.789 |
| E8-5                            | 6.9649 | 7.5601  | 9.2726   | 56.174  | 63.833 |
| $E8-5^{zz}$                     | 6.8918 | 7.5601  | 9.2726   | 56.036  | 63.955 |

**Figure 5.7** Modal shapes of laminated beams. (a) Mode IV of Tab.5.7. (b) Mode V of Tab.5.7. (c) Mode IV of Tab.5.8 and Mode V of Table 5. (d) Mode IV of Tab.5.9.



**Table 5.9** Dimensionless natural frequencies,  $L/b=5$ ,  $[0/90]$  beam.

|  | Mode I | Mode II | Mode III | Mode IV | Mode V | DOFs |
|--|--------|---------|----------|---------|--------|------|
| TE <sub>23</sub> (Giunta et al., 2013) | 4.9375 | 6.4603  | 9.0852   | 33.718  | 50.640 | 900  |
| EBBT                                   | 6.0083 | 10.102  | -        | 57.186  | -      | 198  |
| FSDT                                   | 5.0738 | 7.5051  | -        | 40.961  | -      | 198  |
| TE2                                    | 5.0551 | 6.9637  | 10.133   | 37.566  | 63.570 | 396  |
| TE2 <sup>zz</sup>                      | 5.0440 | 6.9632  | 10.133   | 36.384  | 59.734 | 462  |
| TE3                                    | 4.9947 | 6.6601  | 9.8334   | 36.165  | 56.963 | 660  |
| TE3 <sup>zz</sup>                      | 4.9942 | 6.6593  | 9.8330   | 33.851  | 53.001 | 726  |
| TE6                                    | 4.9462 | 6.5042  | 9.1550   | 34.153  | 51.527 | 1848 |
| TE6 <sup>zz</sup>                      | 4.9384 | 6.5037  | 9.1550   | 33.583  | 50.799 | 1914 |
| E1-6                                   | 5.0212 | 7.2227  | 10.235   | 37.565  | 60.877 | 858  |
| E1-6 <sup>zz</sup>                     | 4.9675 | 7.2227  | 10.235   | 34.202  | 55.170 | 924  |
| E1-4 <sub>4</sub>                      | 4.9406 | 6.5510  | 9.2579   | 34.519  | 52.504 | 1518 |
| E1-4 <sub>4</sub> <sup>zz</sup>        | 4.9360 | 6.5503  | 9.2582   | 33.564  | 51.480 | 1584 |
| E2-4                                   | 4.9438 | 7.2222  | 10.237   | 33.957  | 54.648 | 1122 |
| E2-4 <sup>zz</sup>                     | 4.9381 | 7.2222  | 10.237   | 33.575  | 54.094 | 1188 |
| E2-3 <sub>3</sub>                      | 4.9407 | 6.6589  | 9.8328   | 33.927  | 53.147 | 1452 |
| E2-3 <sub>3</sub> <sup>zz</sup>        | 4.9362 | 6.6586  | 9.8328   | 33.590  | 52.615 | 1518 |
| E3-4                                   | 4.9435 | 7.2154  | 10.237   | 33.920  | 54.637 | 1650 |
| E3-4 <sup>zz</sup>                     | 4.9381 | 7.2154  | 10.237   | 33.586  | 54.087 | 1716 |
| E4-4                                   | 5.0065 | 7.2356  | 10.238   | 34.749  | 57.884 | 594  |
| E4-4 <sup>zz</sup>                     | 5.0059 | 7.2356  | 10.236   | 33.775  | 54.525 | 660  |
| E5-3                                   | 4.9458 | 7.2220  | 10.235   | 34.525  | 55.559 | 858  |
| E5-3 <sup>zz</sup>                     | 4.9407 | 7.2220  | 10.235   | 33.613  | 54.139 | 924  |
| E6-2 <sup>zz</sup>                     | 4.9518 | 7.2222  | 10.235   | 33.758  | 54.156 | 660  |
| E6-1 <sub>3</sub> <sup>zz</sup>        | 4.9411 | 6.6586  | 9.8328   | 33.713  | 52.624 | 990  |
| E8-5                                   | 6.9649 | 7.5601  | 9.2726   | 56.174  | 63.833 | 1386 |
| E8-5 <sup>zz</sup>                     | 6.8918 | 7.5601  | 9.2726   | 56.036  | 63.955 | 1452 |

## Sandwich Beams

The sandwich beams considered in this section consist of structural face sheets (f) bonded to a core (c), whose material properties are:

$$\begin{aligned} E_f &= 68.9 \text{ GPa} & E_c &= 179.14 \text{ MPa} & G_f &= 26.5 \text{ GPa} & G_c &= 68.9 \text{ MPa} \\ \rho_f &= 2687.3 \text{ Kg/m}^3, & \rho_c &= 119.69 \text{ Kg/m}^3 \end{aligned}$$

The first structure has a rectangular cross-section, where the thicknesses of the face sheets and the core are  $h_f = 0.40624$  and  $h_c = 6.3475$  mm, respectively and the width is  $b = 25.4$  mm. The length is  $L = 1.2187$  m and both ends are clamped. The results of the present theories are compared with those found in (Howson and Zare, 2005) and (Banerjee et al., 2007) where the experimental values of Raville (Raville et al., 1961) were reported, and with those of a three-dimensional FEM solution (see Tab.5.10). The modal shapes considered are the first seven flexural modes on yz-plane and the first two torsional ones. For the latter, it is worth noting that expansions TE7 and E8 give best results whereas, for the flexural modes, the use of zig-zag term shows significant improvement of the related solutions, especially for TE1 and TE2.

The subsequent analysis involves a clamped-clamped sandwich beam with a lower length-to-thickness ratio ( $L/h = 5$ ). The cross-section is square and the ratio  $h_c/h_f$  is assumed to be equal to 8. The first eight dimensionless frequencies are compared with a FEM solution, obtained using solid elements *HEX20* of *MSC NASTRAN*®. The values are presented in the following dimensionless form:

$$\bar{\omega} = \frac{L^2}{b} \sqrt{\frac{\rho_f}{G_f}} \omega$$

Given the results in Tab.5.11, it is clear that only refined kinematic models are able to describe the dynamic behaviour of this kind of structure. First of all, the improvements introduced by the zig-zag term are more evident than in the previous example. For instance, considering the natural dimensionless frequency computed with TE7, the relative error decreases from 12% to 0.05%. Further improvements concerns the VII frequency, whose related modal shape (Fig.5.8-b) involves an important deformation of the soft core of the structure. In this case, considering the expansion TE3, the error decreases approximately from 100% to 8%. For the remaining frequencies, the increase of the terms in the displacement field is more effective than the use of the zig-zag function. Mode VI (Fig.5.8-a) is an antisymmetric modal shape with respect to the yz-plane while mode VIII (Fig.5.8-c) involves again the deformation of the core. Furthermore, it must be noted that the last mode is detected only by the miscellaneous expansions (in which the order of Taylor-type polynomials is at least 4) and by E9 expansions. The latter provide appreciable results for all eight dimensionless frequencies with a low number of degrees of freedom.

### 5.2.2 The Functionally Graded cylinder

In this example a FGM sandwich cylinder is studied. Referring to Fig.5.9, the ratios length-to-mean radius ( $L/R$ ) and mean radius-to-thickness ( $R/h$ ) are assumed to be 5 and 10, respectively. The FG core is perfectly bonded to two isotropic faces, whose thicknesses are equal to  $h_1 = h_3 = 0.1h$ . The Young's modulus and the mass density of the core vary according to the following formula:

**Table 5.10** Comparative results for the natural frequencies [Hz] of a fixed-fixed sandwich beam.

|                                 | Mode I | Mode II | Mode III | Mode IV | Mode V | Mode VI | Mode VII | Mode VIII <sup>a</sup> | Mode IX <sup>a</sup> |
|---------------------------------|--------|---------|----------|---------|--------|---------|----------|------------------------|----------------------|
| Exper.(Raville et al., 1961)    | -      | -       | 185.50   | 280.30  | 399.40 | 535.20  | 680.70   | -                      | -                    |
| (Howson and Zare, 2005)         | 34.597 | 93.100  | 177.16   | 282.78  | 406.33 | 544.33  | 693.79   | -                      | -                    |
| DSM (Banerjee et al., 2007)     | 34.342 | 91.385  | 171.69   | 270.36  | 383.27 | 506.88  | 638.39   | -                      | -                    |
| FEM 3D                          | 34.817 | 93.676  | 178.20   | 284.37  | 408.44 | 546.93  | 696.77   | 298.97                 | 598.88               |
| EBBT                            | 35.435 | 97.668  | 191.44   | 316.41  | 472.62 | 660.22  | 879.53   | -                      | -                    |
| FSDT                            | 35.418 | 97.563  | 191.09   | 315.53  | 470.77 | 656.75  | 873.58   | -                      | -                    |
| TE1                             | 35.418 | 97.563  | 191.09   | 315.53  | 470.77 | 656.75  | 873.58   | 1137.9                 | 2276.4               |
| TE1 <sup>zz</sup>               | 34.696 | 93.340  | 177.58   | 283.34  | 407.15 | 545.29  | 695.54   | 1137.9                 | 2276.2               |
| TE2                             | 35.628 | 98.146  | 192.24   | 317.45  | 473.70 | 660.93  | 879.34   | 868.71                 | 1738.2               |
| TE2 <sup>zz</sup>               | 34.895 | 93.853  | 178.50   | 284.72  | 408.97 | 547.56  | 698.15   | 867.86                 | 1733.9               |
| TE4                             | 35.229 | 95.567  | 183.61   | 296.26  | 430.67 | 583.69  | 753.05   | 365.31                 | 729.38               |
| TE4 <sup>zz</sup>               | 34.958 | 94.016  | 178.79   | 285.15  | 409.54 | 548.28  | 698.98   | 365.10                 | 728.80               |
| TE7                             | 35.043 | 94.518  | 180.35   | 288.72  | 416.26 | 559.41  | 715.75   | 338.73                 | 674.84               |
| TE7 <sup>zz</sup>               | 34.951 | 93.977  | 178.75   | 285.08  | 409.43 | 548.10  | 698.71   | 338.66                 | 674.66               |
| E2-5                            | 36.580 | 98.428  | 187.31   | 298.96  | 429.71 | 575.72  | 734.49   | 1139.0                 | 2276.7               |
| E2-5 <sup>zz</sup>              | 36.480 | 97.893  | 185.70   | 295.37  | 423.06 | 564.85  | 718.28   | 1139.0                 | 2276.7               |
| E4-3 <sub>3</sub>               | 35.071 | 94.825  | 181.53   | 291.72  | 422.05 | 569.17  | 731.09   | 867.43                 | 1733.4               |
| E4-3 <sub>3</sub> <sup>zz</sup> | 34.982 | 93.836  | 178.28   | 284.84  | 409.31 | 547.13  | 701.35   | 867.23                 | 1732.1               |
| E5-2 <sub>2</sub>               | 34.984 | 94.369  | 180.10   | 288.38  | 415.81 | 558.89  | 715.30   | 867.36                 | 1733.2               |
| E5-2 <sub>2</sub> <sup>zz</sup> | 34.892 | 93.845  | 178.48   | 284.69  | 408.92 | 547.50  | 698.05   | 867.26                 | 1732.6               |
| E8-5                            | 35.904 | 96.692  | 184.19   | 294.30  | 423.49 | 568.03  | 725.46   | 369.69                 | 730.74               |
| E8-5 <sup>zz</sup>              | 35.809 | 96.184  | 182.65   | 290.86  | 417.11 | 557.55  | 709.79   | 369.55                 | 730.50               |

(–): expansions with zig-zag term.

<sup>a</sup>: torsional mode.

**Table 5.11** Comparative results for the dimensionless frequencies of a fixed-fixed sandwich beam with  $L/b = 5$ .

|                                 | Mode I <sup>a</sup> | Mode II <sup>a</sup> | Mode III <sup>c</sup> | Mode IV <sup>a</sup> | Mode V <sup>b</sup> | Mode VI* | Mode VII* | Mode VIII* | DOFs  |
|---------------------------------|---------------------|----------------------|-----------------------|----------------------|---------------------|----------|-----------|------------|-------|
| FEM 3D                          | 2.0308              | 4.3874               | 6.6198                | 7.3156               | 7.7618              | 10.461   | 10.603    | 10.673     | 68277 |
| EBBT                            | 14.373              | 35.786               | -                     | 61.872               | 9.4430              | -        | -         | -          | 279   |
| FSDT                            | 10.367              | 22.116               | -                     | 35.737               | 7.9752              | -        | -         | -          | 279   |
| TE1                             | 10.368              | 22.116               | 15.118                | 35.737               | 7.9752              | 30.233   | 88.065    | -          | 279   |
| TE1 <sup>zz</sup>               | 2.1069              | 4.6382               | 15.091                | 7.8576               | 8.8191              | 29.965   | 19.171    | -          | 372   |
| TE3                             | 3.4838              | 7.1907               | 13.763                | 11.428               | 7.9197              | 31.614   | 21.320    | -          | 930   |
| TE3 <sup>zz</sup>               | 2.1030              | 4.6302               | 15.896                | 7.8498               | 7.9198              | 29.963   | 11.445    | -          | 1023  |
| TE6                             | 2.2745              | 4.8532               | 7.0430                | 7.9833               | 7.8136              | 12.188   | 12.294    | 12.252     | 2604  |
| TE6 <sup>zz</sup>               | 2.0325              | 4.3951               | 6.9046                | 7.3309               | 7.8136              | 11.402   | 11.032    | 12.252     | 2697  |
| TE7                             | 2.2687              | 4.8424               | 6.9949                | 7.9683               | 7.8120              | 11.847   | 11.971    | 12.234     | 3348  |
| TE7 <sup>zz</sup>               | 2.0319              | 4.3923               | 6.8579                | 7.3222               | 7.8120              | 11.162   | 10.805    | 12.234     | 3441  |
| E1-3 <sub>4</sub>               | 2.1753              | 4.6773               | 7.7521                | 7.4934               | 7.8533              | 13.142   | 12.132    | 19.943     | 1953  |
| E1-3 <sub>4</sub> <sup>zz</sup> | 2.0386              | 4.4155               | 7.4239                | 7.3746               | 7.8533              | 12.710   | 11.330    | 19.943     | 2046  |
| E5-3 <sup>zz</sup>              | 2.0434              | 4.4312               | 15.079                | 7.4039               | 8.1020              | 29.911   | 11.378    | -          | 1302  |
| E8-5 <sup>zz</sup>              | 2.1022              | 4.6233               | 7.3860                | 7.8253               | 7.8947              | 12.540   | 17.664    | -          | 2046  |
| E9-2 <sup>zz</sup>              | 2.0343              | 4.3993               | 8.6952                | 7.3368               | 8.0206              | 12.493   | 13.502    | 16.390     | 2418  |
| E9-3 <sup>zz</sup>              | 2.0317              | 4.3914               | 6.9315                | 7.3220               | 7.8181              | 11.114   | 10.823    | 12.284     | 4650  |
| E9-4 <sup>zz</sup>              | 2.0299              | 4.3850               | 6.7931                | 7.3065               | 7.8055              | 10.924   | 10.760    | 11.698     | 7626  |

(-): expansions with zig-zag term.

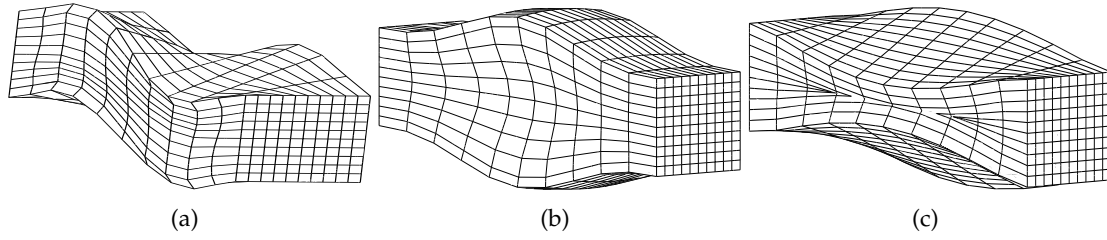
- : results not provided by the model.

\*: mode in Fig.5.

<sup>a</sup>: bending mode in  $z$  - direction.

<sup>b</sup>: bending mode in  $x$  - direction.

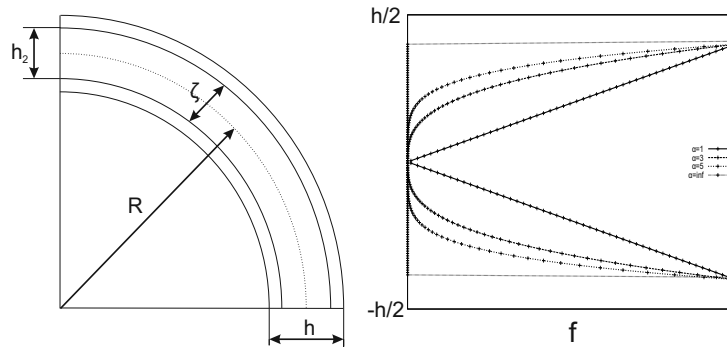
<sup>c</sup>: torsional mode.



**Figure 5.8** Modal shapes of sandwich beam. (a) Mode VI (b) Mode VII (c) Mode VIII.

$$f = (f_1 - f_2) \left( \frac{2|\zeta|}{h_2} \right)^\alpha + f_2 \quad (5.4)$$

where  $h_2$  is the core thickness,  $\zeta$  is the generic coordinate ( $-\frac{h_2}{2} < \zeta < \frac{h_2}{2}$ ) and the exponent  $\alpha$  assumes the values 3, 5 and  $\infty$ . For this case,  $E_1$  and  $E_2$  are assumed to be 70 and 380 GPa, whereas the mass densities  $\rho_1, \rho_2$  are 2702 and 3800 Kg/m<sup>3</sup>, respectively. The Poisson's ratio  $\nu$  is constant and equal to 0.3. For the sake of clarity, Fig.5.9 shows how the properties  $E$  and  $\rho$  change along the thickness. The results in Tab.5.12 are reported in terms of non-dimensional frequency parameter  $\bar{\omega} = \omega h \sqrt{\frac{\rho_2}{E_2}}$  and compared with those presented in (Wu and Kuo, 2013), in which a semi-analytical solution based on the finite layer method was developed for the simply-supported cylindrical structures. As expected, accuracy improves when the displacement field is enriched. The two graphs of Fig.5.10 illustrate how the percentage error  $\left( \frac{(\bar{\omega} - \bar{\omega}_{ref}) \times 100}{\bar{\omega}_{ref}} \right)$  varies with respect to the number of degrees of freedom (DoF). For both values of the exponent  $\alpha$  (3 and  $\infty$ ), it is possible to note that higher order models are needed to reach acceptable results for all modes of deformation and indeed, by using either the tenth-order Taylor expansion or the E9-4 series, errors remain lower than 5%.



**Figure 5.9** Cylinder cross-section and variations of the generic property  $f$  along the thickness.

### 5.2.3 Thick metallic and composite cylinders

This section presents the results obtained from free-vibration analyses carried out on moderately thick shells at standstill. Later on, the same structures will be considered in order to evaluate the effects of the rotational speed on their frequencies.

First of all, a thick isotropic cylinder has been considered. Its material properties are: Young's modulus  $E = 207$  GPa, Poisson's ratio  $\nu = 0.28$  and density  $\rho = 7860$  kg/m<sup>3</sup>. Both ends are

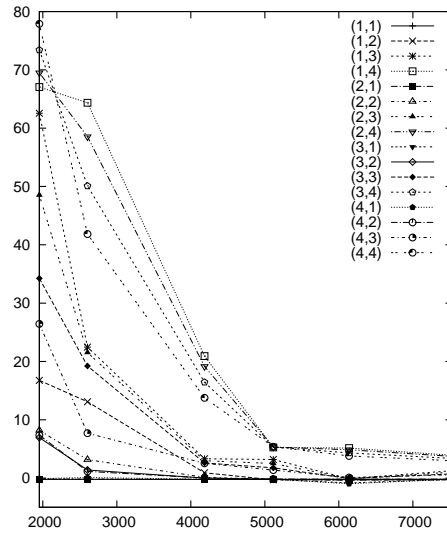
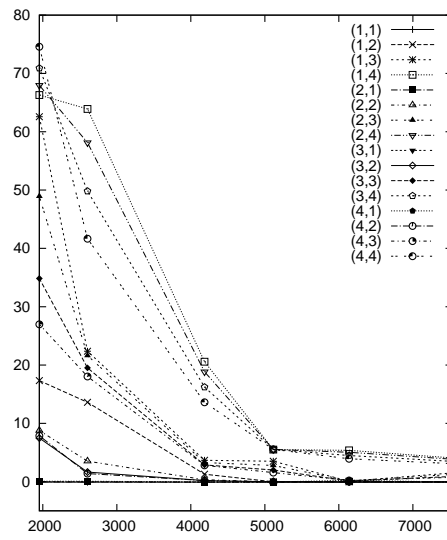
free (F-F) and the length, the mean radius and the thickness of the structure are 0.254 m, 0.09525 m and 0.0381 m, respectively. This cylinder was previously studied in (Wang et al., 1998) where its three-dimensional mode shapes were accurately classified. The accuracy of the 1D-CUF elements is evaluated, comparing the natural frequencies with those obtained using theoretical as well as experimental approaches. It is possible to observe in Tab.5.13 that both TE expansions (TE8 and TE9) yield results very close to the 3D solution for all considered modes.

Secondly, a thick cylinder made of orthotropic material is studied. The mechanical properties are:  $E_{11} = 211$  GPa,  $E_{22} = E_{33} = 24.1$  GPa,  $G_{12} = G_{13} = G_{23} = 6.9$  GPa,  $\nu_{12} = \nu_{13} = \nu_{23} = 0.36$  and  $\rho = 1967$  kg/m<sup>3</sup>. The thickness-to-diameter and the length-to-diameter ratios are  $\alpha = 0.1667$  and  $k = 2.5$ , respectively, with the mean diameter equal to 0.1 m. The considered structure is simply supported and clamped at its ends and the layers have the same thickness. The natural frequencies computed with various Taylor-like expansions, and for different lamination schemes, are listed in Tab.5.14. For the purpose of comparison, in the last column the results obtained with the DIQUASPAB software are shown. It should be noted that the beam element TE8 provides accurate results, the maximum percentage error is approximately 7% for the radial mode  $n = 3$  and lamination scheme  $[0^\circ/0^\circ]$ . Furthermore, it is well known that the lamination scheme strongly affects the order of appearance of the modes. For instance, when the ply angle is equal to  $90^\circ$ , the shell-like modes occur at higher frequencies than those relating to the bending and extensional deformations, contrary to what happens for  $0^\circ$ .

**Table 5.12** Non-dimensional frequencies parameter  $\bar{\omega}$  for a sandwich FGM cylinder.

| $\alpha$ | (m,n) | TE5      | TE6      | TE8      | TE9      | TE10     | E9-4     | LD <sub>33</sub> (Wu and Kuo, 2013) |
|----------|-------|----------|----------|----------|----------|----------|----------|-------------------------------------|
| 3        | (1,1) | 0.015361 | 0.015361 | 0.015360 | 0.015360 | 0.015357 | 0.015360 | 0.015396                            |
|          | (1,2) | 0.012379 | 0.011991 | 0.010700 | 0.010572 | 0.010572 | 0.010570 | 0.010603                            |
|          | (1,3) | 0.037004 | 0.027878 | 0.023520 | 0.023487 | 0.022739 | 0.023055 | 0.022764                            |
|          | (1,4) | 0.069182 | 0.068064 | 0.050075 | 0.043592 | 0.043541 | 0.043031 | 0.041413                            |
|          | (2,1) | 0.037971 | 0.037967 | 0.037962 | 0.037962 | 0.037905 | 0.037962 | 0.038044                            |
|          | (2,2) | 0.024680 | 0.023535 | 0.022843 | 0.022764 | 0.022762 | 0.022762 | 0.022814                            |
|          | (2,3) | 0.041048 | 0.033589 | 0.028462 | 0.028326 | 0.027616 | 0.027917 | 0.027637                            |
|          | (2,4) | 0.075675 | 0.070775 | 0.053192 | 0.047022 | 0.046805 | 0.046301 | 0.044643                            |
|          | (3,1) | 0.055409 | 0.055400 | 0.055380 | 0.055379 | 0.055032 | 0.055380 | 0.055498                            |
|          | (3,2) | 0.040272 | 0.038185 | 0.037661 | 0.037585 | 0.037582 | 0.037582 | 0.037661                            |
|          | (3,3) | 0.049534 | 0.043988 | 0.037851 | 0.037544 | 0.036867 | 0.037163 | 0.036897                            |
|          | (3,4) | 0.088012 | 0.076183 | 0.059094 | 0.053461 | 0.052947 | 0.052460 | 0.050754                            |
|          | (4,1) | 0.066753 | 0.066750 | 0.066674 | 0.066672 | 0.066173 | 0.066672 | 0.066811                            |
|          | (4,2) | 0.055221 | 0.052033 | 0.051449 | 0.051352 | 0.051346 | 0.051348 | 0.051442                            |
|          | (4,3) | 0.061498 | 0.057241 | 0.049866 | 0.049301 | 0.048679 | 0.048912 | 0.048636                            |
|          | (4,4) | 0.106066 | 0.084568 | 0.067819 | 0.062819 | 0.061887 | 0.061404 | 0.059622                            |
| 5        | (1,1) | 0.014674 | 0.014673 | 0.014673 | 0.014673 | 0.014672 | 0.014673 | 0.014728                            |
|          | (1,2) | 0.012008 | 0.011629 | 0.010369 | 0.010235 | 0.010236 | 0.010233 | 0.010286                            |
|          | (1,3) | 0.036057 | 0.027151 | 0.022950 | 0.022915 | 0.022153 | 0.022466 | 0.022212                            |
|          | (1,4) | 0.067267 | 0.066243 | 0.048739 | 0.042526 | 0.042470 | 0.041959 | 0.040374                            |
|          | (2,1) | 0.036278 | 0.036273 | 0.036269 | 0.036269 | 0.036271 | 0.036269 | 0.036397                            |
|          | (2,2) | 0.023750 | 0.022622 | 0.021939 | 0.021854 | 0.021853 | 0.021853 | 0.021938                            |
|          | (2,3) | 0.039935 | 0.032645 | 0.027691 | 0.027547 | 0.026820 | 0.027119 | 0.026885                            |
|          | (2,4) | 0.073364 | 0.068858 | 0.051761 | 0.045849 | 0.045614 | 0.045111 | 0.043497                            |
|          | (3,1) | 0.052961 | 0.052952 | 0.052930 | 0.052929 | 0.052798 | 0.052928 | 0.053116                            |
|          | (3,2) | 0.038704 | 0.036648 | 0.036124 | 0.036042 | 0.036013 | 0.036039 | 0.036172                            |
|          | (3,3) | 0.048072 | 0.042645 | 0.036698 | 0.036371 | 0.035691 | 0.035972 | 0.035768                            |
|          | (3,4) | 0.084994 | 0.074066 | 0.057468 | 0.052071 | 0.051528 | 0.051034 | 0.049390                            |
|          | (4,1) | 0.063862 | 0.063859 | 0.063776 | 0.063774 | 0.063708 | 0.063774 | 0.063998                            |
|          | (4,2) | 0.053099 | 0.049964 | 0.049370 | 0.049266 | 0.049261 | 0.049262 | 0.049433                            |
|          | (4,3) | 0.059573 | 0.055414 | 0.048263 | 0.047658 | 0.046905 | 0.047268 | 0.047061                            |
|          | (4,4) | 0.102079 | 0.082142 | 0.065901 | 0.061110 | 0.060092 | 0.059630 | 0.057938                            |
| $\infty$ | (1,1) | 0.013160 | 0.013160 | 0.013159 | 0.013159 | 0.013156 | 0.013159 | 0.013170                            |
|          | (1,2) | 0.010916 | 0.010570 | 0.009423 | 0.009302 | 0.009301 | 0.009301 | 0.009304                            |
|          | (1,3) | 0.032905 | 0.024767 | 0.020979 | 0.020948 | 0.020254 | 0.020543 | 0.020239                            |
|          | (1,4) | 0.061290 | 0.060407 | 0.044434 | 0.038878 | 0.038829 | 0.038352 | 0.036854                            |
|          | (2,1) | 0.032540 | 0.032536 | 0.032532 | 0.032532 | 0.032554 | 0.032531 | 0.032555                            |
|          | (2,2) | 0.021441 | 0.020402 | 0.019777 | 0.019701 | 0.019701 | 0.019700 | 0.019710                            |
|          | (2,3) | 0.036398 | 0.029727 | 0.025240 | 0.025120 | 0.024475 | 0.024741 | 0.024440                            |
|          | (2,4) | 0.066673 | 0.062774 | 0.047179 | 0.041896 | 0.041694 | 0.041225 | 0.039696                            |
|          | (3,1) | 0.047522 | 0.047513 | 0.047494 | 0.047494 | 0.047568 | 0.047493 | 0.047526                            |
|          | (3,2) | 0.034903 | 0.033008 | 0.032532 | 0.032459 | 0.032459 | 0.032458 | 0.032473                            |
|          | (3,3) | 0.043723 | 0.038750 | 0.033355 | 0.033070 | 0.032458 | 0.032712 | 0.032423                            |
|          | (3,4) | 0.076978 | 0.067483 | 0.052349 | 0.047532 | 0.047063 | 0.046611 | 0.045042                            |
|          | (4,1) | 0.057351 | 0.057349 | 0.057277 | 0.057277 | 0.057450 | 0.057275 | 0.057310                            |
|          | (4,2) | 0.047908 | 0.045023 | 0.044493 | 0.044403 | 0.044401 | 0.044400 | 0.044410                            |
|          | (4,3) | 0.054100 | 0.050294 | 0.043797 | 0.043275 | 0.042698 | 0.042926 | 0.042609                            |
|          | (4,4) | 0.092171 | 0.074786 | 0.059985 | 0.055719 | 0.054865 | 0.054426 | 0.052797                            |

 $m$ : half-waves in  $y$ -direction. $n$ : half-waves in circumferential direction.

(a)  $\alpha = 3$ (b)  $\alpha = \infty$ **Figure 5.10** Percentage error versus the number of degrees of freedom.



**Table 5.13** Frequencies [Hz] at standstill of the thick isotropic cylinder.

| Modal description |   |   | Frequency [Hz] |       |                      |                       |                         |
|-------------------|---|---|----------------|-------|----------------------|-----------------------|-------------------------|
| Mode Type         | n | m | TE8            | TE9   | 2D(Guo et al., 2001) | 3D(Wang et al., 1998) | Exp.(Wang et al., 1998) |
| Pure Radial       | 2 | 0 | 2746           | 2606  | 2573                 | 2594                  | 2570                    |
| Radial Shearing   | 2 | 1 | 3101           | 2993  | 2950                 | 2989                  | 2962                    |
| Axial Bending     | 1 | 1 | 6233           | 6230  | 6239                 | 6240                  | 6286                    |
| Global Torsion    | 0 | 1 | 6313           | 6313  | 6320                 | 6291                  | -                       |
| Radial Shearing   | 2 | 2 | 6484           | 6433  | -                    | 6503                  | 6536                    |
| Gobal Bending     | 1 | 2 | 7038           | 7033  | 7134                 | 7109                  | 7104                    |
| Radial Shearing   | 3 | 1 | 7574           | 7502  | -                    | 7219                  | 7100                    |
| Extensional       | 0 | 0 | 8111           | 8108  | 8096                 | 8081                  | 8149                    |
| Extensional       | 0 | 1 | 8558           | 8551  | -                    | 8567                  | 8615                    |
| Extensional       | 0 | 2 | 8821           | 8815  | -                    | 8845                  | 8886                    |
| Longitudinal      | 0 | 0 | 11272          | 11269 | 11266                | 11460                 | 11536                   |
| Circumferential   | 1 | 1 | 12673          | 12673 | 12653                | 12647                 | 12616                   |

-: result not provided.

**Table 5.14** Frequencies [Hz] at standstill of the C-S composite cylinder.

| Mode Type                  | n | m | TE6    | TE7    | TE8                       | DQM*   |
|----------------------------|---|---|--------|--------|---------------------------|--------|
| $0^\circ/0^\circ$          |   |   |        |        |                           |        |
| Extensional                | 0 | 0 | 10167. | 10160. | 10160. <sup>(0.04)</sup>  | 10155. |
| Bending                    | 1 | 1 | 2885.8 | 2862.4 | 2862.3 <sup>(0.16)</sup>  | 2857.5 |
| Radial                     | 2 | 1 | 3945.9 | 3827.8 | 3781.6 <sup>(3.94)</sup>  | 3638.0 |
| Radial                     | 2 | 2 | 6498.8 | 6431.5 | 6272.5 <sup>(2.16)</sup>  | 6139.7 |
| Radial                     | 3 | 1 | 8343.4 | 8305.3 | 7978.6 <sup>(7.07)</sup>  | 7451.6 |
| Radial                     | 3 | 2 | 9933.8 | 9855.8 | 9589.1 <sup>(6.79)</sup>  | 8978.8 |
| Torsion                    | 1 | 0 | 3745.9 | 3745.9 | 3745.9 <sup>(0.00)</sup>  | 3745.9 |
| $30^\circ/-30^\circ$       |   |   |        |        |                           |        |
| Extensional                | 0 | 0 | 5863.8 | 5849.7 | 5849.4 <sup>(0.26)</sup>  | 5833.7 |
| Bending                    | 1 | 1 | 5227.5 | 5206.1 | 5204.3 <sup>(-0.42)</sup> | 5226.5 |
| Radial                     | 2 | 1 | 4485.7 | 4308.1 | 4348.3 <sup>(2.84)</sup>  | 4227.9 |
| Radial                     | 2 | 2 | 8315.0 | 8261.9 | 8168.3 <sup>(0.51)</sup>  | 8126.1 |
| Radial                     | 3 | 1 | 8790.1 | 8757.7 | 8459.4 <sup>(5.93)</sup>  | 7985.5 |
| Radial                     | 3 | 2 | 10736. | 10679. | 10429. <sup>(5.10)</sup>  | 9922.5 |
| Torsion                    | 1 | 0 | 8331.1 | 8322.9 | 8316.3 <sup>(-0.82)</sup> | 8385.8 |
| $45^\circ/-45^\circ$       |   |   |        |        |                           |        |
| Extensional                | 0 | 0 | 3637.7 | 3631.4 | 3630.3 <sup>(0.06)</sup>  | 3627.9 |
| Bending                    | 1 | 1 | 4458.7 | 4446.6 | 4444.3 <sup>(-0.05)</sup> | 4446.9 |
| Radial                     | 2 | 1 | 4397.3 | 4307.9 | 4271.5 <sup>(2.71)</sup>  | 4158.6 |
| Radial                     | 2 | 2 | 7390.9 | 7311.1 | 7229.4 <sup>(0.70)</sup>  | 7178.9 |
| Radial                     | 3 | 1 | 9649.5 | 9522.0 | 9236.3 <sup>(5.03)</sup>  | 8793.7 |
| Radial                     | 3 | 2 | 11110. | 10989. | 10719. <sup>(4.98)</sup>  | 10210. |
| Torsion                    | 1 | 0 | 9725.8 | 9710.4 | 9706.9 <sup>(-0.66)</sup> | 9771.4 |
| $90^\circ/90^\circ$        |   |   |        |        |                           |        |
| Extensional                | 0 | 0 | 3522.9 | 3520.9 | 3520.8 <sup>(0.54)</sup>  | 3501.8 |
| Bending                    | 1 | 1 | 2309.3 | 2302.4 | 2302.3 <sup>(0.17)</sup>  | 2298.2 |
| Radial                     | 2 | 1 | 6344.7 | 5998.0 | 5986.4 <sup>(3.65)</sup>  | 5775.2 |
| Radial                     | 2 | 2 | 7198.1 | 6909.6 | 6851.2 <sup>(2.46)</sup>  | 6686.3 |
| Radial                     | 3 | 1 | 14641. | 14620. | 12836. <sup>(6.20)</sup>  | 12086. |
| Radial                     | 3 | 2 | 15043. | 14918. | 13213. <sup>(6.17)</sup>  | 12444. |
| Torsion                    | 1 | 0 | 3745.9 | 3745.9 | 3745.9 <sup>(0.00)</sup>  | 3745.9 |
| $0^\circ/90^\circ/0^\circ$ |   |   |        |        |                           |        |
| Extensional                | 0 | 0 | 8735.9 | 8733.6 | 8731.0 <sup>(0.00)</sup>  | 8731.7 |
| Bending                    | 1 | 1 | 2869.0 | 2845.3 | 2845.2 <sup>(0.20)</sup>  | 2839.4 |
| Radial                     | 2 | 1 | 4135.6 | 4044.6 | 3998.8 <sup>(4.02)</sup>  | 3843.9 |
| Radial                     | 2 | 2 | 6597.1 | 6543.0 | 6384.8 <sup>(2.31)</sup>  | 6240.5 |
| Radial                     | 3 | 1 | 8989.6 | 8951.6 | 8621.3 <sup>(6.83)</sup>  | 8069.7 |
| Radial                     | 3 | 2 | 10465. | 10382. | 10104. <sup>(6.75)</sup>  | 9465.0 |
| Torsion                    | 1 | 0 | 3745.9 | 3745.9 | 3745.9 <sup>(0.00)</sup>  | 3745.9 |

\*: first-order shear deformation shell theory.

In brackets, the relative errors:  $(100 \cdot (\text{DQM} - \text{TE8})) / \text{DQM}$ .

## 5.3 The spinning structures

### 5.3.1 Classical beam theories

In order to assess the theory presented in Sec.3.2, shafts made of isotropic material are here considered. The boundary conditions and the ratio between the cross-section dimensions are assumed to be problem's parameters. Ten finite elements are used for the discretization along the  $y$  – axis. With the purpose of enabling a general application of results, they are presented in non-dimensional form by adopting the following non-dimensional natural frequency and spinning speed parameters:

$$\omega^* = \frac{\omega}{\omega_0}, \quad \Omega^* = \frac{\Omega}{\omega_0}, \quad \omega_0 = \sqrt{\frac{E J_{xx} E J_{yy}}{\rho A L^4}},$$

where  $J_{xx}$ ,  $J_{zz}$  are the moments of inertia in the two principal planes,  $E$  is the Young's Modulus,  $A$  the area of the cross-section and  $L$  the length of the beam. In the first example, the dynamic behavior of a beam with  $J_{xx} = J_{zz}$  which is subjected to different boundaries conditions is investigated. Equation (5.5) is exploited for comparison purposes and the relative results are reported in Fig.5.11.

$$\omega_{ref} = \lambda^2 \sqrt{\frac{EI}{\rho A L^4}} \mp \Omega \quad (5.5)$$

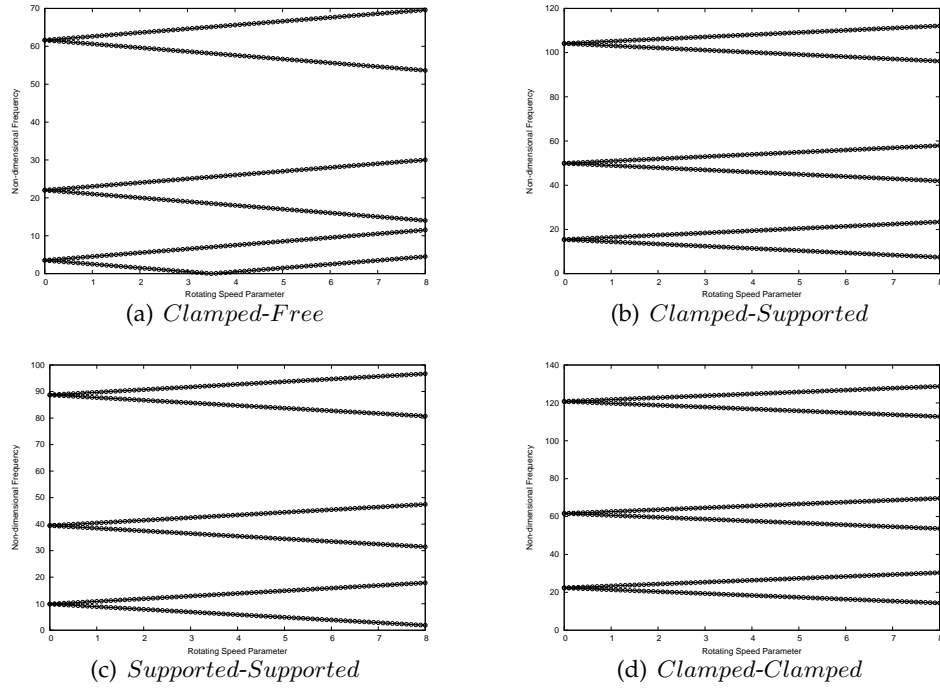
$\lambda$  is a coefficient depending on the boundary conditions.

As easily shown, the results derived by the Unified Formulation match those of the analytical solutions for every type of boundary condition. Since the effects of shear deformation and rotatory inertia are considered to be small for this type of rotor, the study has been conducted using Euler-Bernoulli theory. In Tab.5.15, some values of non-dimensional natural frequency at different spinning speed parameters are compared with those obtained by Banerjee *et al* in (Banerjee and Su, 2004) for the cantilever beam. It should be noted that the frequencies are very close to those of the reference, especially the value of the critical speeds.

**Table 5.15** Frequency ratios of a cantilever spinning beam with  $J_{xx} = J_{zz}$ .

| $\Omega^*$ | Theory                  | $\omega_1^*$ | $\omega_2^*$ | $\omega_3^*$ | $\omega_4^*$ | $\omega_5^*$ | $\omega_6^*$ |
|------------|-------------------------|--------------|--------------|--------------|--------------|--------------|--------------|
| 0          | (Banerjee and Su, 2004) | 3.516        | 3.516        | 22.034       | 22.034       | 61.697       | 61.967       |
|            | Present                 | 3.510        | 3.510        | 21.992       | 21.992       | 61.551       | 61.551       |
| 2          | (Banerjee and Su, 2004) | 1.516        | 5.516        | 20.034       | 24.034       | 59.697       | 63.697       |
|            | Present                 | 1.510        | 5.510        | 19.995       | 23.992       | 59.559       | 63.552       |
| 3.51       | (Banerjee and Su, 2004) | 0            | 7.016        | 18.534       | 25.534       | 58.197       | 65.197       |
|            | Present                 | 0            | 7.020        | 18.486       | 25.500       | 58.052       | 65.059       |
| 4          | (Banerjee and Su, 2004) | -            | 7.516        | 18.034       | 26.034       | 57.697       | 65.697       |
|            | Present                 | -            | 7.509        | 17.996       | 25.990       | 57.563       | 65.548       |

In the same work, Banerjee *et al.* presented the results related to beams with rectangular cross-sections. Figure 5.12 shows how the first two flexural natural frequencies are affected by the



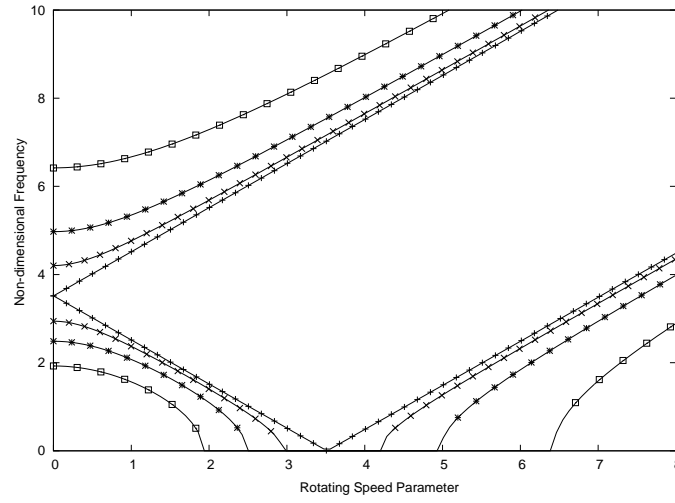
**Figure 5.11** The variation of non-dimensional natural frequencies with spinning speed parameter for a beam ( $J_{xx}$ ,  $J_{zz}$ ) subjected to several boundary conditions ('lines': Eq.5.5; '⊙': CUF).

ratio between the main cross-section dimensions,  $b/h$ , when the rotating speed is changing. The reference solutions are obtained using Eq.5.6, where  $\tilde{\omega}_x$  and  $\tilde{\omega}_z$  are the natural frequencies at standstill.

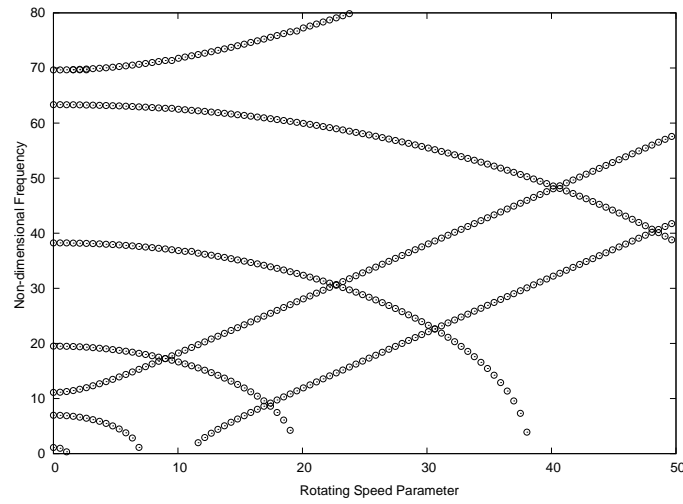
$$\omega_{x,z} = \sqrt{\frac{1}{2} \{ -(\tilde{\omega}_x^2 + \tilde{\omega}_z^2 + 2\Omega^2) \pm \sqrt{(\tilde{\omega}_x^2 - \tilde{\omega}_z^2)^2 + 8\Omega^2(\tilde{\omega}_x^2 + \tilde{\omega}_z^2)} \}} \quad (5.6)$$

It is clear that the behaviors of the frequencies in the two principal planes are very different. In fact, the first frequency decreases until it becomes zero, whereas the second continues to increase. It is important to note that, for these values of *aspect – ratio*, the flexural normal modes in the two planes are alternated. On the contrary, when the width becomes much smaller than the height, the relationship for the frequency ratios undergoes an observable change. For instance, if  $b = 0.1h$ , the first two mode frequency ratios in the  $x$  direction are smaller than the first one in  $z$  direction. This difference is shown in Fig. 5.13, where the first seven frequency ratios are presented. For the last case, in Tab.5.16 the results obtained with CUF are compared with the numerical values of frequency ratios presented in ((Banerjee and Su, 2004)).

As the effects of shear deformation can be disregarded when the shaft is very thin, but if this approximation is no longer valid, other structural models are needed. In the following test case, a moderate thick cylindrical beam simply supported is analyzed. The length and the radius are assumed to be equal to 10 m and 0.25 m, respectively. The material has the Young's modulus equal to 210 GPa whereas the Poisson coefficient and the density are 0.3 and 7800kg/m<sup>3</sup>. The results in Tab.5.17 are compared with those provided by Curti *et al.* (Curti *et al.*, 1992), who proposed an analytical procedure based on DSM including the effects of the shear deformation of the shaft. It is evident that, also in this case, the proposed results strongly agree with the reference solutions.



**Figure 5.12** The first two frequency ratios  $w_1^*$  and  $w_2^*$  as functions of the spinning speed ratio for various values of aspect-ratio (‘-’: Eq.5.6; ‘+’:  $b/h = 1$ ; ‘x’:  $b/h = 0.7$ ; ‘\*’:  $b/h = 0.5$ ; ‘□’:  $b/h = 0.3$ ).



**Figure 5.13** The first seven frequency ratios  $w_1^* - w_7^*$  as functions of the spinning speed ratio for aspect-ratio equal to 0.1.

**Table 5.16** Frequency ratios of a cantilever spinning beam with rectangular cross-section ( $b = 0.1h$ ).

| $\Omega^*$ | <i>Theory</i>           | $\omega_1^*$ | $\omega_2^*$ | $\omega_3^*$ | $\omega_4^*$ |
|------------|-------------------------|--------------|--------------|--------------|--------------|
| 0          | (Banerjee and Su, 2004) | 1.1119       | 6.9680       | 11.119       | 19.511       |
|            | Present                 | 1.1114       | 6.9658       | 11.119       | 19.508       |
| 1.11       | (Banerjee and Su, 2004) | 0            | 6.8623       | 11.307       | 19.473       |
|            | Present                 | 0            | 6.8732       | 11.284       | 19.475       |
| 2          | (Banerjee and Su, 2004) | 0            | 6.6637       | 11.632       | 19.404       |
|            | Present                 | 0            | 6.6611       | 11.633       | 19.401       |
| 4          | (Banerjee and Su, 2004) | -            | 5.6679       | 12.900       | 19.080       |
|            | Present                 | -            | 5.6638       | 12.901       | 19.077       |
| 6          | (Banerjee and Su, 2004) | -            | 3.4910       | 14.536       | 18.530       |
|            | Present                 | -            | 3.4818       | 14.538       | 18.526       |
| 7.1        | (Banerjee and Su, 2004) | -            | 0            | 15.526       | 18.118       |
|            | Present                 | -            | 0            | 15.524       | 18.116       |
| 8          | (Banerjee and Su, 2004) | -            | -            | 16.3367      | 17.7348      |
|            | Present                 | -            | -            | 16.3399      | 17.7310      |

**Table 5.17** Forward critical speeds predicted by CUF for the test case of a simply supported shaft.

| Mode Number | (Curti et al., 1992) | <i>EBBM</i> | <i>FSDT</i> |
|-------------|----------------------|-------------|-------------|
| 1           | 63.869               | 63.699      | 63.550      |
| 2           | 253.769              | 255.001     | 252.449     |
| 3           | 564.741              | 573.379     | 564.299     |
| 4           | 989.050              | 1019.405    | 988.400     |
| 5           | 1516.866             | 1599.400    | 1519.259    |

*EBBM* stands for Euler-Bernoulli Beam Model.

*FSDT* stands for First-order Shear Deformation Theory.

### 5.3.2 Higher order beam theories for spinning composite shafts

#### Cylindrical composite shaft

This section shows the results related to analyses carried out on composite thin-walled structures. The first case concerns the dynamic study of hollow rotating cylinders whose material and geometrical features are summarised in Tab.5.18. The shaft has been discretised with seven 4-nodes beam elements and it is supported by one bearing at each end, whose stiffnesses  $k_{xx}$  and  $k_{zz}$  are 1740 GN/m. The first critical speeds obtained with 1D-CUF theories are shown in Tab.5.19 and compared with those found in (Chang et al., 2004), adopting the  $[90^\circ/45^\circ/-45^\circ/0^\circ_6/90^\circ]$  lamination scheme. The results for both materials are within the interval of reference speeds, and the differences between the three theories (TE2, TE3 and TE4) are essentially negligible. Furthermore, the effects of the aspect ratio for the graphite-epoxy shaft are also investigated. As expected, when the structure becomes thicker, the critical speeds occur at higher values, and the differences between the three expansions increase. In fact, when the aspect ratio is equal to 2, TE3 and TE4 lead to similar results, whereas TE2 overestimates the critical speed of 10%. In addition, considering the same material, the effects due to the lamination scheme have been evaluated and the results are shown in Tab.5.21. The increasing the lamination angle with respect to the longitudinal direction reduces the shaft stiffness and causes a consequent reduction in the critical speeds. Some evident discrepancies can be observed between the refined beam models and those found in literature. In fact, when  $\theta$  is equal to  $15^\circ$ ,  $30^\circ$  or  $45^\circ$ , the present results are closer to speeds obtained with the shell theory SST than those of classical beam models. In order to show the accuracy of the CUF elements, natural frequencies of the boron-epoxy shaft (Tab.5.18), computed with TE elements, are listed in Tab.5.22 and compared with the results found in (Boukhalfa, 2011), where a first-order shear beam theory with a torsion function was proposed, and with those obtained using Di.Qu.M.A.S.P.A.B., a free software based on the Differential Quadrature Method (GDQ) for shells and plates. At least the third-order theory (TE3) is necessary to obtain accurate results for bending as well as torsional modes. Moreover, it is worth noting that remarkable differences exist between the first natural frequencies of (Boukhalfa, 2011) and those obtained with the other approaches, especially for  $\theta$  equal to  $15^\circ$ ,  $30^\circ$ ,  $45^\circ$  and  $60^\circ$ . The use of a constant shear correction factor could be the reason for these important discrepancies, since the shear effects are closely related to the lamination scheme.

#### Thin-walled composite shaft

In the following numerical illustrations, box beams with thin walls, whose dimensions are shown in Tab.5.18, are studied. The results are displayed for different cross-section height-to-width ratios ( $R = b/c$ , Fig.5.14) and lamination angles ( $\theta$ ), in terms of non-dimensional frequency ( $\bar{f} = f/f_n$ ), as a function of the speed parameter ( $\bar{\Omega} = \Omega/f_n$ ). The value  $f_n$  is the natural frequency of a cantilever graphite-epoxy box beam with  $R = 1$  and  $\theta = 90^\circ$  obtained with a 2D finite element solution ( $f_n = 52.649$  [Hz]). Figure 5.15 shows how the first two frequency ratios vary with the rotational speed for  $\theta$  equal to  $0^\circ$  and  $90^\circ$ , using various expansions. As expected, the theory order does not affect the trend of the branches, with the exception that curves start at lower values when the displacement field is enriched, and these differences are more evident for the case in which  $\theta$  is equal to  $0^\circ$ . This fact demonstrates that classical models (and ad-hoc theories!) cannot guarantee the same accuracy for all lamination angles, thus confirming that a simple method to conceive increasingly accurate theories could be a very useful

**Table 5.18** Material and geometrical features of composite shaft.

| Dimensions                  | Cylinders                        | Box beam                            |
|-----------------------------|----------------------------------|-------------------------------------|
| Thickness ( $h$ ) [m]       | 0.001321                         | 0.01016                             |
| Width ( $c$ ) [m]           | 0.1269*                          | 0.1016                              |
| Length ( $L$ ) [m]          | 2.470                            | 1.016                               |
| Materials                   | Boron-epoxy (Chang et al., 2004) | Graphite-epoxy (Chang et al., 2004) |
| $E_{11}$ [Gpa]              | 211.0                            | 139.0                               |
| $E_{22}$ [Gpa]              | 24.1                             | 11.0                                |
| $G_{23}$ [Gpa]              | 6.90                             | 3.78                                |
| $G_{31}$ [Gpa]              | 6.90                             | 6.05                                |
| $G_{12}$ [Gpa]              | 6.90                             | 6.05                                |
| $\rho$ [kg/m <sup>3</sup> ] | 1967.0                           | 1578.0                              |
| $\nu_{12}$                  | 0.360                            | 0.313                               |

\*: mean diameter.

**Table 5.19** Critical speeds (rpm) of composite shafts for  $[90^\circ/45^\circ/-45^\circ/0^\circ_6/90^\circ]$ .

| Theory                    | Boron-epoxy | Graphite-epoxy |
|---------------------------|-------------|----------------|
| SST (Chang et al., 2004)  | 5872        | 5349           |
| EBBT (Chang et al., 2004) | 5919        | 5302           |
| BTBT (Chang et al., 2004) | 5788        | 5113           |
| FSDT (Chang et al., 2004) | 5762        | 5197           |
| TE2                       | 5808        | 5256           |
| TE3                       | 5766        | 5232           |
| TE4                       | 5724        | 5220           |

*SST*: Sanders shell theory.*EBBT*: Euler-Bernoulli beam theory.*BTBT*: Bresse-Timoshenko beam theory.*FSDT*: First-order shear deformation theory.

tool. Moreover, in order to evaluate the effect due to the ply angle, the first two frequency ratios are shown in Fig.5.16, adopting the TE2 and TE6 expansions. Critical speeds occur at different values, depending on the lamination angle, and maximum and minimum velocities occur for  $\theta = 0^\circ$  and  $\theta = 90^\circ$ , respectively. Similar results were obtained by Librescu *et al.* in (Song and Librescu, 1997a). In addition, it must be noted that different lamination schemes determine the occurrence of couplings between different deformation modes. For instance, in Fig.5.14, two different configurations  $[\theta_T/\theta_L/\theta_B/\theta_R]$  are studied: *Case I*  $[45^\circ/-45^\circ/-45^\circ/45^\circ]$  and *Case II*  $[45^\circ/-45^\circ/45^\circ/-45^\circ]$ . The frequencies ( $f$ ) and damping ( $D$ ) ratios are computed as functions of the rotating speed using a number of displacement models (Fig.5.17 and Fig.5.18). Classical theories (EBBT and FSDT, Fig.5.17-a) provide qualitatively similar results for the first configuration, but do not predict any instability range. Nevertheless, with the first-order shear deformation theory, a veering of the fourth, fifth and sixth branches can be detected. TE3 and TE6 expansions, instead, correspond to dramatic changes with three instability fields appear-



**Table 5.20** Critical speeds (rpm) of graphite-epoxy shaft for several length-to-diameter ratios.

| Theory                    | L/D    |       |      |
|---------------------------|--------|-------|------|
|                           | 2      | 5     | 15   |
| SST (Chang et al., 2004)  | 112400 | 41680 | 8585 |
| EBBT (Chang et al., 2004) | 329600 | 76820 | 9072 |
| BTBT (Chang et al., 2004) | 176300 | 54830 | 8543 |
| FSDT (Chang et al., 2004) | 176300 | 55706 | 8527 |
| TE2                       | 240900 | 65580 | 8760 |
| TE3                       | 206400 | 61200 | 8580 |
| TE4                       | 206400 | 61200 | 8580 |

**Table 5.21** Critical speeds (rpm) of composite shaft for various lamination angles.

| Theory                    | Lamination |      |      |      |      |      |      |
|---------------------------|------------|------|------|------|------|------|------|
|                           | 0°         | 15°  | 30°  | 45°  | 60°  | 75°  | 90°  |
| SST (Chang et al., 2004)  | 5527       | 4365 | 3308 | 2386 | 2120 | 2020 | 1997 |
| EBBT (Chang et al., 2004) | 6425       | 5393 | 4269 | 3171 | 2292 | 1885 | 1813 |
| FSDT (Chang et al., 2004) | 6072       | 5331 | 4206 | 3124 | 2284 | 1890 | 1816 |
| TE1                       | 6270       | 4770 | 3120 | 2280 | 1950 | 1860 | 1860 |
| TE2                       | 6270       | 4710 | 3120 | 2280 | 1950 | 1860 | 1860 |
| TE3                       | 6060       | 4170 | 2760 | 2160 | 1935 | 1860 | 1830 |
| TE4                       | 6060       | 4140 | 2730 | 2145 | 1920 | 1830 | 1800 |

ing within the considered speed interval. This configuration involves a dominant twist-motion therefore models that are able to describe bending-torsional coupling are needed. In addition, when the expansion order is increased, instability thresholds are predicted at lower speed values. On the contrary, for the second case, the most important effect is due to shear and, for this reason, the shearable beam model leads to comparable results with those obtained with higher-order theories. For both schemes, the gyroscopic coupling markedly affects the system, causing several veerings of frequency branches.

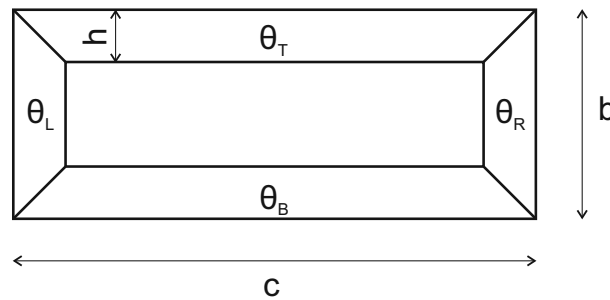
In the last illustrative example, a rectangular box beam ( $R = 0.5$ ) is analysed with the purpose of considering another kind of anisotropy. Hereafter, the reference frequency  $f_n$  is equal to 25.976[Hz]. Figure 5.19-a shows the first two frequency ratios varying with the speed parameter when  $\theta$  is  $0^\circ$  or  $15^\circ$ . We compare four different theories and, for both cases, the curves move towards lower and lower values when the displacement model is enriched. For  $\theta = 0^\circ$  (thin lines), TE3 branches almost overlap TE6 theory branches, while, discrepancies between these models clearly increase when  $\theta$  is equal to  $15^\circ$  (bold lines). Therefore, in order to ensure a good accuracy for other lamination schemes, the TE6 expansion is selected and the results are shown in Fig.5.19-b. It is worth noting that the curves follow similar paths regardless of the lamination. In fact, after a veering of the backward frequency branch, two eigenvalues interact and instability occurs. Furthermore, when  $\theta$  increases, the instability threshold appears at lower and lower speeds, while the range of the instability field becomes smaller, ultimately reaching

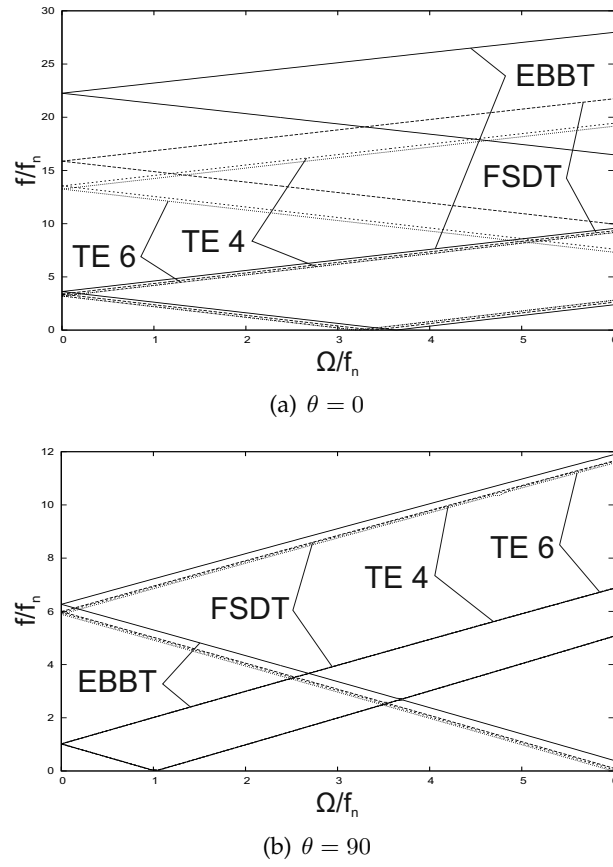
**Table 5.22** Frequencies at standstill of boron-epoxy shaft.

| Lam. | Mode     | (Boukhalifa, 2011) | GDQ    | TE4    | TE3    | TE2    | TE1*   | FSDT*  | EBBT*  |
|------|----------|--------------------|--------|--------|--------|--------|--------|--------|--------|
| 0°   | 1° Flex. | 108.23             | 109.23 | 109.24 | 109.24 | 113.95 | 113.95 | 113.95 | 119.44 |
|      | 2° Flex. | -                  | 357.13 | 357.15 | 357.19 | 403.38 | 403.37 | 403.37 | 475.41 |
|      | 1° Tors. | -                  | 379.13 | 379.14 | 379.14 | 379.14 | 379.14 | -      | -      |
| 15°  | 1° Flex. | 94.86              | 69.57  | 69.83  | 70.05  | 81.73  | 81.76  | 81.76  | 108.39 |
|      | 2° Flex. | -                  | 261.23 | 262.15 | 262.81 | 309.33 | 309.47 | 309.47 | 431.46 |
|      | 1° Tors. | -                  | 412.23 | 412.79 | 412.79 | 413.60 | 412.60 | -      | -      |
| 30°  | 1° Flex. | 76.71              | 45.82  | 46.05  | 46.60  | 52.41  | 52.33  | 52.33  | 74.39  |
|      | 2° Flex. | -                  | 180.30 | 181.11 | 183.10 | 205.85 | 205.16 | 205.16 | 296.12 |
|      | 1° Tors. | -                  | 519.80 | 519.98 | 519.98 | 520.63 | 519.80 | -      | -      |
| 45°  | 1° Flex. | 59.21              | 37.85  | 37.98  | 38.48  | 39.57  | 39.54  | 39.54  | 43.62  |
|      | 2° Flex. | -                  | 150.13 | 150.55 | 152.51 | 156.95 | 156.30 | 156.30 | 173.62 |
|      | 1° Tors. | -                  | 638.00 | 639.43 | 639.44 | 691.04 | 663.24 | -      | -      |
| 60°  | 1° Flex. | 45.84              | 36.49  | 36.59  | 36.74  | 36.75  | 36.75  | 36.75  | 36.89  |
|      | 2° Flex. | -                  | 144.11 | 144.30 | 145.10 | 145.35 | 145.14 | 145.14 | 146.84 |
|      | 1° Tors. | -                  | 534.03 | 540.09 | 540.10 | 675.61 | 586.18 | -      | -      |
| 75°  | 1° Flex. | 40.74              | 38.33  | 38.46  | 38.58  | 38.99  | 39.00  | 39.00  | 39.58  |
|      | 2° Flex. | -                  | 149.60 | 149.75 | 150.42 | 153.16 | 153.21 | 153.21 | 157.54 |
|      | 1° Tors. | -                  | 415.73 | 419.76 | 419.71 | 460.93 | 429.83 | -      | -      |
| 90°  | 1° Flex. | 40.10              | 39.80  | 39.95  | 40.47  | 40.70  | 40.79  | 40.79  | 41.03  |
|      | 2° Flex. | -                  | 154.05 | 154.18 | 156.07 | 159.34 | 159.71 | 159.71 | 163.33 |
|      | 1° Tors. | -                  | 379.14 | 379.14 | 379.14 | 379.14 | 379.14 | -      | -      |

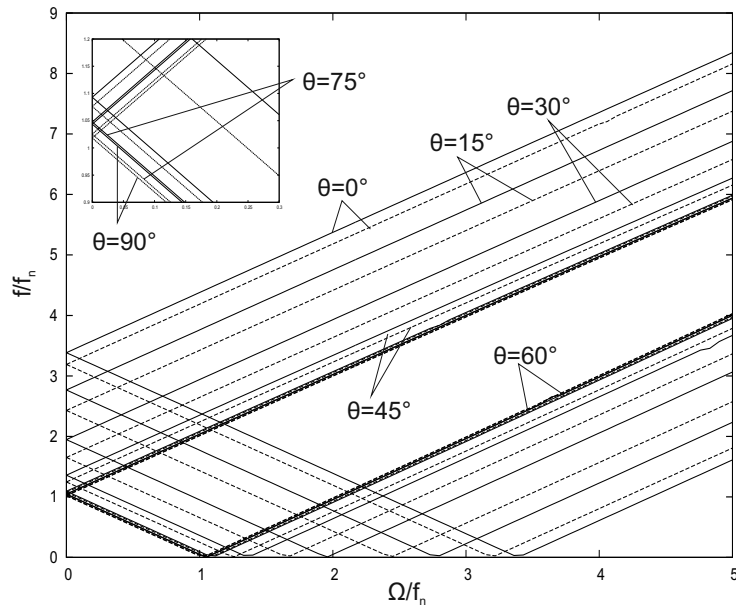
\*: Poisson locking correction activated.

a minimum for  $\theta = 90^\circ$ . The effects of ply angles on damping are shown in Fig.5.19-c.

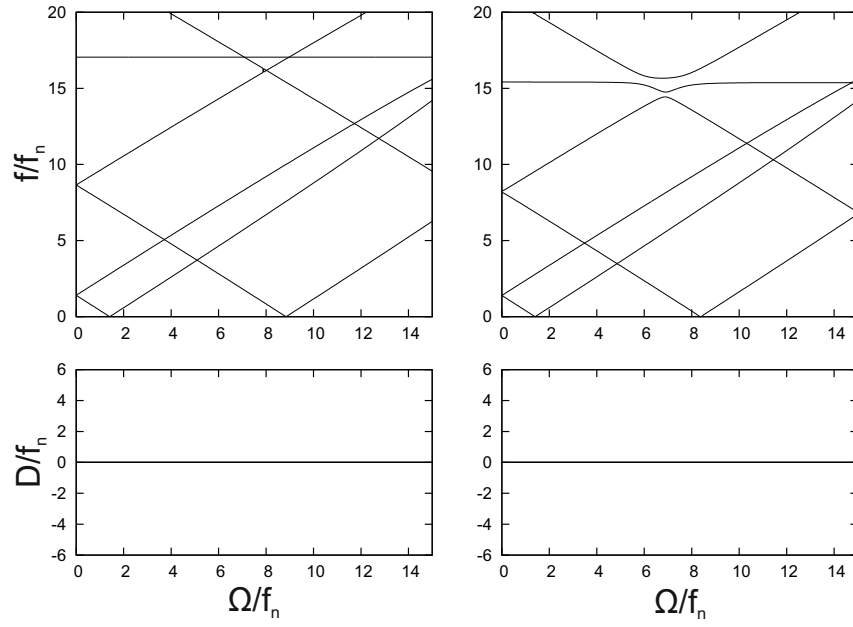
**Figure 5.14** Sketch of the box beam.



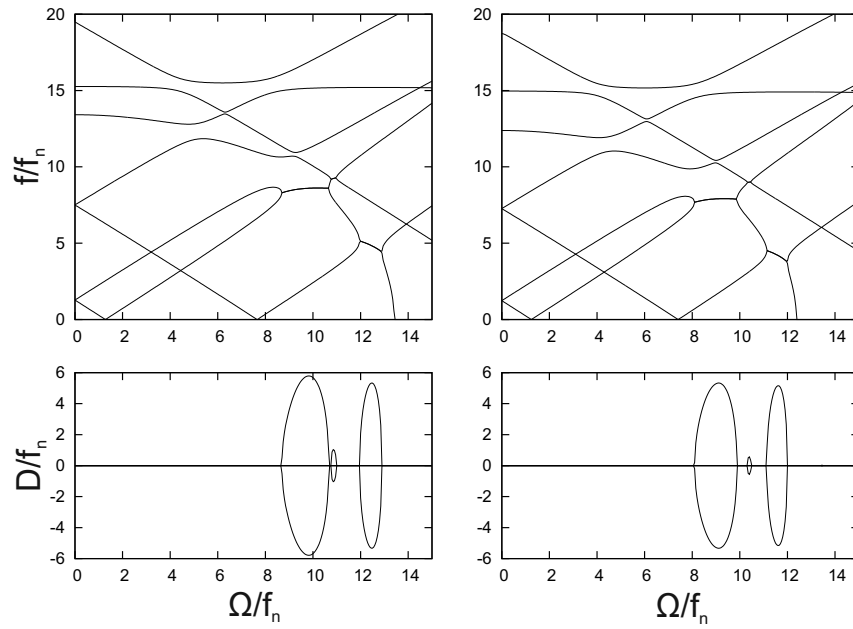
**Figure 5.15** Dependency of frequency ratios upon the speed parameter for the cantilever box beam ( $R = 1$ ).



**Figure 5.16** Dependency of frequency ratio upon the speed parameter for various values of ply angles ( $R = 1$ ), '—': TE2, '- - -': TE6.

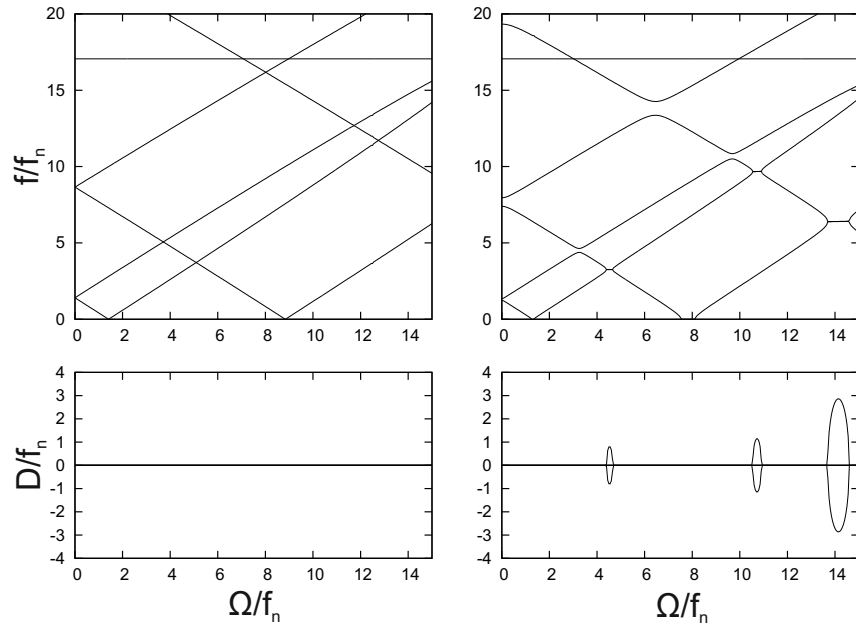


(a) EBBT, FSDT

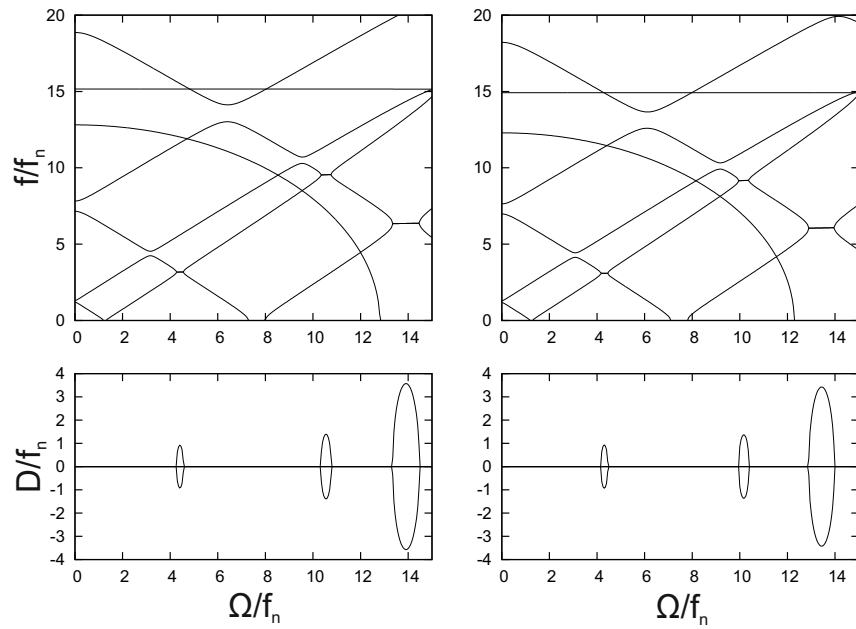


(b) TE3, TE6

**Figure 5.17** Dependency of frequency ratio and damping upon the speed parameter for a cantilever box beam ( $R = 1$  and Case I).

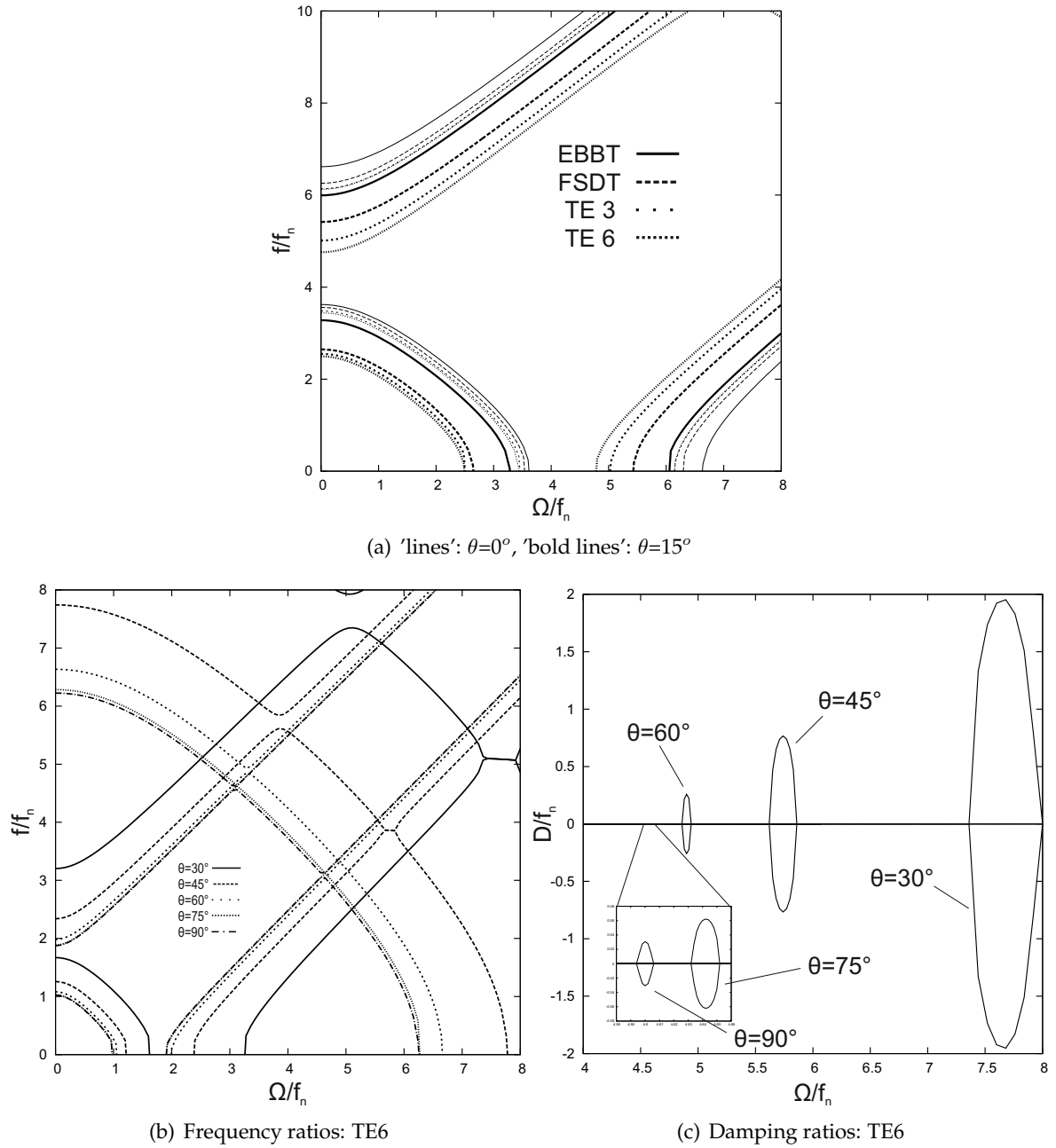


(a) EBBT, FSDT



(b) TE3, TE6

**Figure 5.18** Dependency of frequency and damping ratios upon the speed parameter for a cantilever box beam ( $R = 1$  and Case II).



**Figure 5.19** Dependency of frequency and damping ratios upon the speed parameter for a cantilever box beam ( $R = 0.5$ ).

### 5.3.3 Highly deformable spinning structures

#### Thin walled cylinders

The following analyses are devoted to the study of the influence of centrifugal stiffness on the dynamic behaviour of thin rotating shells (see Sec.3.2.2). To this end, isotropic and composite cylinders with different dimension ratios, boundary conditions and lamination schemes have been studied.

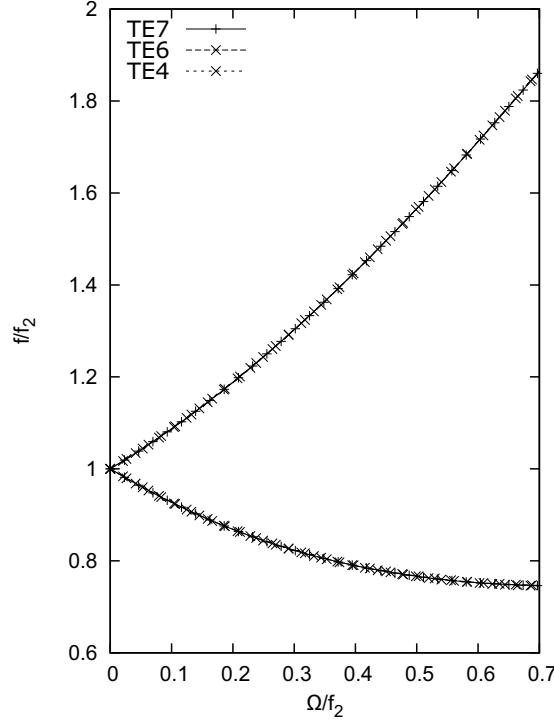
In the first numerical computation, a cylinder with thickness-to-diameter ratio  $\alpha = 0.02$  and length-to-diameter ratio  $k = 5$ . The material is isotropic with Young's modulus ( $E'$ ), Poisson's ratio ( $\nu'$ ) and density equal to 207.0 GPa, 0.3 and 7860 Kg/m<sup>3</sup>, respectively. Table 5.23 shows the frequencies at standstill for different dimension ratios ( $\alpha$ = thickness/diameter and  $k$ =length/diameter with diameter equal to 1) obtained with Taylor-like expansions of fourth (TE4), fifth (TE5), sixth (TE6) and seventh (TE7) order. For comparison purposes, the last column shows the frequencies obtained using the DiQuMASPAB code. As expected, increasing the order of the theory, results converge to reference values for all considered circumferential wave numbers ( $n'$ ) and boundary conditions.

**Table 5.23** Frequencies [Hz] at standstill of the isotropic cylinder for various  $\alpha$  and  $k$ .

| n  | TE4   | TE5   | TE6   | TE7   | DQM   |
|--|-------|-------|-------|-------|-------|
| Supported-Supported, $\alpha = 0.02$ and $k = 5$ |       |       |       |       |       |
| 1  | 101.2 | 101.2 | 101.2 | 101.2 | 101.2 |
| 2  | 96.29 | 76.25 | 75.57 | 64.61 | 64.61 |
| 3  | 289.4 | 252.1 | 189.0 | 188.1 | 152.2 |
| Clamped-Clamped, $\alpha = 0.02$ and $k = 5$     |       |       |       |       |       |
| 1  | 171.4 | 171.4 | 171.4 | 171.4 | 170.8 |
| 2  | 115.5 | 99.35 | 98.63 | 90.48 | 90.17 |
| 3  | 300.7 | 254.5 | 192.3 | 191.2 | 155.9 |
| Clamped-Clamped, $\alpha = 0.005$ and $k = 5$    |       |       |       |       |       |
| 2  | 75.37 | 73.90 | 73.84 | 73.18 | 72.73 |
| 3  | 178.3 | 72.94 | 59.86 | 59.63 | 52.50 |
| 4  | -     | 215.8 | 122.6 | 92.91 | 75.54 |
| Clamped-Clamped, $\alpha = 0.005$ and $k = 3$    |       |       |       |       |       |
| 2  | 168.4 | 167.5 | 167.4 | 167.1 | 166.6 |
| 3  | 319.1 | 112.0 | 104.0 | 103.6 | 99.26 |
| 4  | -     | 340.9 | 134.1 | 107.7 | 92.38 |

For the supported-supported condition and  $n'$  equal to 2 ( $\alpha = 0.02$  and  $k=5$ ), Fig.5.20 shows how the frequency ratio ( $f/f_n$ ) changes with respect to the speed parameter ( $\Omega/f_n$ ) using the TE4, TE6 and TE7 expansions. In spite of the natural frequencies in the non-rotating state are different, it is interesting to note that the three displacement theories predict same variations of backward and forward non-dimensional frequencies.

To evaluate the effects of boundary conditions, cylinders with supported-supported and clamped-clamped ends are considered. The dimensionless frequencies obtained with the TE7 theory are



**Figure 5.20** Dependency of frequency ratio upon the speed parameter for various TE expansions; '...': TE4, '- -': TE6, '—': TE7 ( $n = 2$ ,  $\alpha = 0.02$  and  $k = 5$ ).

shown in Fig.5.21-a and Fig.5.21-b, respectively.

Also in this case the non-dimensional results are nearly identical despite the different frequencies in the non-rotating state (see Tab.5.23). The graphs also show the frequency parameters derived from the following analytical solutions

$$\frac{f}{f_n} = \frac{2n}{n^2 + 1}(\Omega/f_n) \pm \sqrt{1 + \frac{n^6 - 2n^4 - 1}{(n^2 + 1)^2}(\Omega/f_n)^2} \quad (5.7)$$

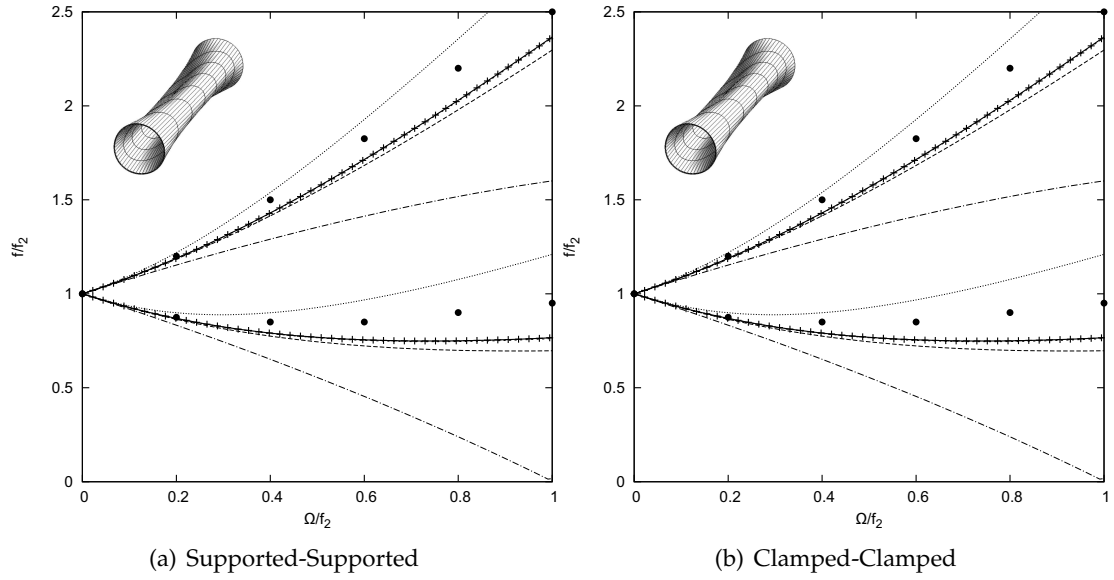
$$\frac{f}{f_n} = \frac{2n}{n^2 + 1}(\Omega/f_n) \pm \sqrt{1 + \frac{n^2(n^2 - 1)^2}{(n^2 + 1)^2}(\Omega/f_n)^2} \quad (5.8)$$

$$\frac{f}{f_n} = \frac{2n}{n^2 + 1}(\Omega/f_n) \pm \sqrt{1 + \frac{n^6 + 3n^2}{(n^2 + 1)^2}(\Omega/f_n)^2} \quad (5.9)$$

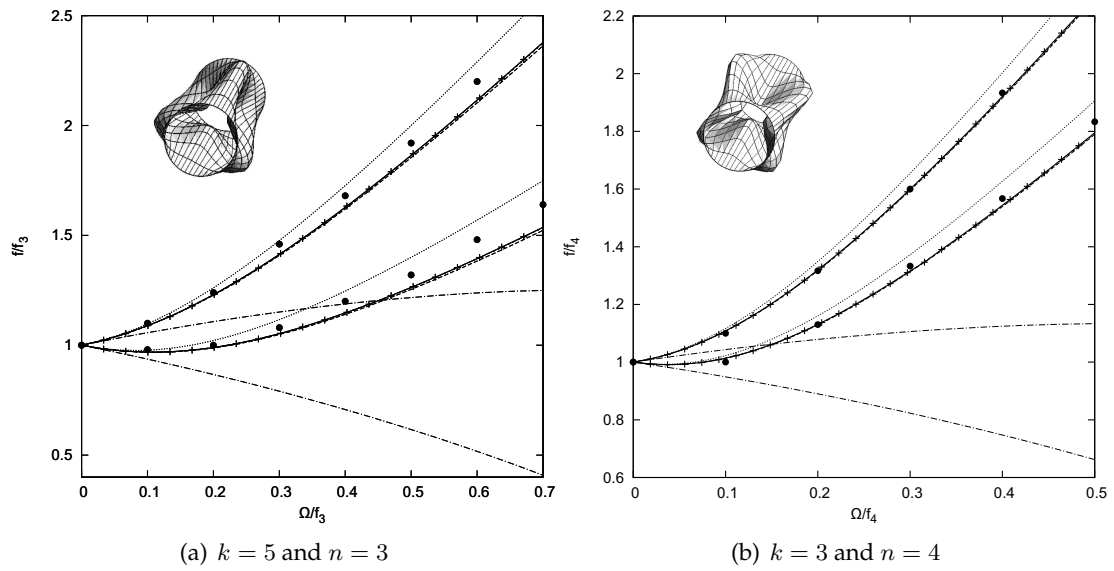
In particular, Eqs.5.7 and 5.8 were proposed by Endo *et al.* in (Endo et al., 1984) for the study of rotating rings. Although the solutions were based on a non-linear shell theory, the expressions present slight differences due to the use of different hypothesis in defining acceleration and strain terms. On the other hand, using the linear approximation, Chen et al. (1993) presented Eq.5.9 for the study of high speed rotating shells. Other reference values (●) are taken from (Endo et al., 1984), where the results presented in (Srinivasan and Lauterbach, 1971) were reported in the dimensionless form. It should be observed that the TE7 results strongly agree with those obtained using Eq.5.8.

Assuming  $\alpha$  equal to 0.005, the minimum frequency for the non-rotating cylinder occurs for  $n = 3$  (see Tab.5.23). Figure 5.22-a is the result for this mode.





**Figure 5.21** Dependency of frequency ratio upon the speed parameter; '—': 5.7, '---': 5.8, '- - -': 5.9, '●': Ref.[7] of (Endo et al., 1984), '...': TE7 w/o  $\sigma_0$ , '+ +': TE7 ( $n = 2$ ,  $\alpha = 0.02$  and  $k = 5$ ).



**Figure 5.22** Dependency of frequency ratio upon the speed parameter; '—': 5.7, '---': 5.8, '- - -': 5.9, '●': Ref.[7] of (Endo et al., 1984), '...': TE7 w/o  $\sigma_0$ , '+ +': TE7 ( $\alpha = 0.005$ ).

Beam results agree with shell solutions, especially with those of Eqs.5.7 and 5.8. The same consideration can be made for the mode with  $n = 4$  ( $\alpha = 0.005$  and  $k = 3$ ), whose frequency variations with respect to the rotational speed are shown in Fig.5.22-b. Table 5.24 lists dimensionless frequencies for five rotational speeds with or without the geometrical stiffening. It should be observed that the contribution due to the hoop tension becomes more evident as the circumferential wave number increases.

|          |                 | $\Omega/f_n$         |                       |                        |                        |                        |                        |
|----------|-----------------|----------------------|-----------------------|------------------------|------------------------|------------------------|------------------------|
|          |                 | 0.0                  | 0.1                   | 0.2                    | 0.3                    | 0.4                    | 0.5                    |
| Forward  |                 |                      |                       |                        |                        |                        |                        |
| $f/f_2$  | with $\sigma_0$ | 1.0                  | 0.93                  | 0.87                   | 0.82                   | 0.79                   | 0.77                   |
|          | w/o $\sigma_0$  | 1.0 <sup>(0.0)</sup> | 0.92 <sup>(1.1)</sup> | 0.83 <sup>(4.6)</sup>  | 0.74 <sup>(9.8)</sup>  | 0.65 <sup>(17.7)</sup> | 0.55 <sup>(28.6)</sup> |
| $f/f_3$  | with $\sigma_0$ | 1.0                  | 0.97                  | 0.99                   | 1.05                   | 1.15                   | 1.26                   |
|          | w/o $\sigma_0$  | 1.0 <sup>(0.0)</sup> | 0.94 <sup>(3.1)</sup> | 0.87 <sup>(12.1)</sup> | 0.79 <sup>(24.8)</sup> | 0.71 <sup>(38.3)</sup> | 0.62 <sup>(50.8)</sup> |
| $f/f_4$  | with $\sigma_0$ | 1.0                  | 1.01                  | 1.13                   | 1.32                   | 1.54                   | 1.79                   |
|          | w/o $\sigma_0$  | 1.0 <sup>(0.0)</sup> | 0.95 <sup>(5.9)</sup> | 0.89 <sup>(21.2)</sup> | 0.82 <sup>(37.9)</sup> | 0.75 <sup>(51.3)</sup> | 0.66 <sup>(63.1)</sup> |
| Backward |                 |                      |                       |                        |                        |                        |                        |
| $f/f_2$  | with $\sigma_0$ | 1.0                  | 1.09                  | 1.19                   | 1.30                   | 1.43                   | 1.57                   |
|          | w/o $\sigma_0$  | 1.0 <sup>(0.0)</sup> | 1.08 <sup>(0.9)</sup> | 1.15 <sup>(3.4)</sup>  | 1.22 <sup>(6.2)</sup>  | 1.29 <sup>(9.8)</sup>  | 1.35 <sup>(14.0)</sup> |
| $f/f_3$  | with $\sigma_0$ | 1.0                  | 1.09                  | 1.23                   | 1.41                   | 1.63                   | 1.86                   |
|          | w/o $\sigma_0$  | 1.0 <sup>(0.0)</sup> | 1.06 <sup>(2.8)</sup> | 1.11 <sup>(9.8)</sup>  | 1.15 <sup>(18.4)</sup> | 1.19 <sup>(27.0)</sup> | 1.22 <sup>(34.4)</sup> |
| $f/f_4$  | with $\sigma_0$ | 1.0                  | 1.11                  | 1.32                   | 1.60                   | 1.92                   | 2.26                   |
|          | w/o $\sigma_0$  | 1.0 <sup>(0.0)</sup> | 1.04 <sup>(6.3)</sup> | 1.08 <sup>(18.2)</sup> | 1.11 <sup>(30.6)</sup> | 1.12 <sup>(41.7)</sup> | 1.13 <sup>(50.0)</sup> |

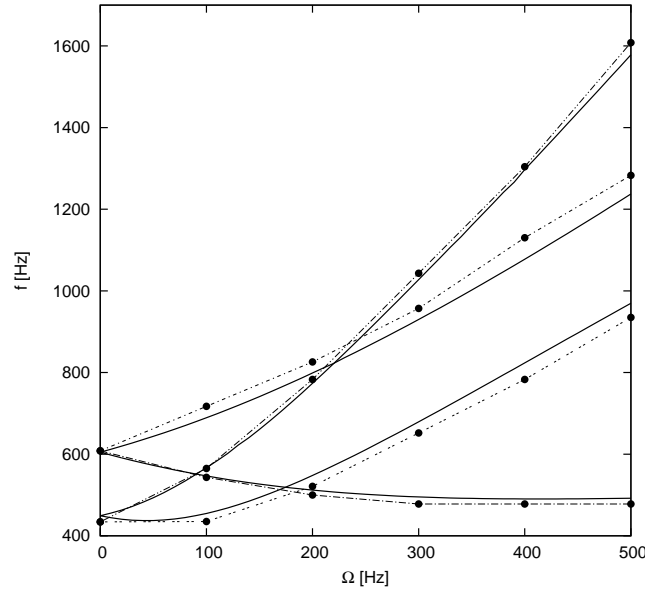
In brackets, relative errors between the two cases.

**Table 5.24** Dimensionless frequencies  $f/f_n$  vs. dimensionless speed  $\Omega/f_n$  with and without the hoop tension for thin cylinders.

Other results are presented in Fig.5.23, where the frequencies of a rotating cantilevered cylinder ( $\alpha = 0.02$  and  $k = 0.5$ ) are shown. In the graph, the branches are related to the modes with  $n = 3$  and  $n = 2$ . The reference solution was taken from (Guo et al., 2002), where a finite element based on a non-linear plant-shell theory was developed. Although there are some slight discrepancies, trends of curves obtained with TE7 expansion are similar to the reference. It is worth noting that the present beam element provides very accurate results for  $\Omega=0$ .

### Moderately thick cylinder

Regarding the isotropic cylinder studied in Sec.5.2.3, the evolutions of a number of frequencies listed in Tab.5.13 as functions of the rotating speed are investigated. Figure 5.24 illustrates how frequency parameters change with the rotational speed for global bending ( $n = 1, m = 2$ ) and



**Figure 5.23** Dependency of frequency ratio upon the speed parameter; '-+-':  $n = 2$  (Guo et al., 2002), '-x-':  $n = 3$  (Guo et al., 2002), '—': TE8 ( $\alpha = 0.02$  and  $k = 0.5$ ).

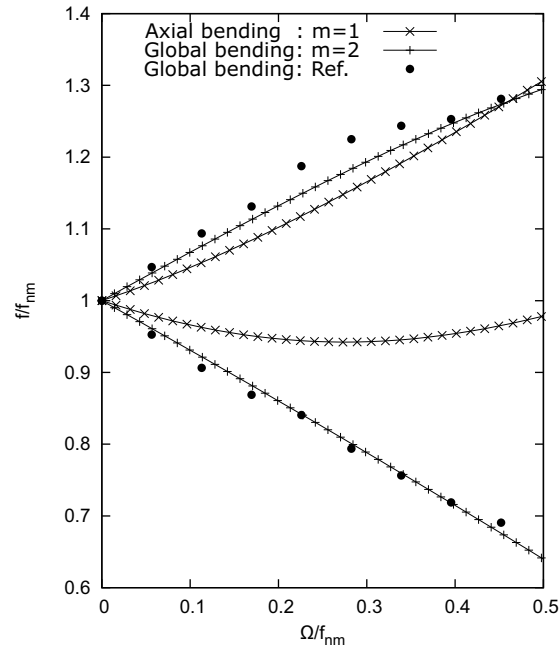
axial bending modes ( $n = 1, m = 1$ ). It should be noted that the non-linear variation of the global bending frequencies predicted by the 1D-CUF elements is quite similar to that presented in (Guo et al., 2001).

Figure 5.25 shows the results for pure radial ( $m = 0$ ) and radial shearing ( $m = 1, 2$ ) modes for  $n$  equal to 2 and 3. It is possible to note that a different longitudinal wave number involves changes in the frequency variations, especially for low values of the circumferential wave number  $n$ . As also stated in (Guo et al., 2001), these differences are not detectable from the above analytical solutions (Eqs.5.7,5.9). Furthermore, Fig.5.26 illustrates the variations of the extensional and torsional frequencies. Although nearly constant, the two extensional frequencies exhibit different behaviour upon the speed variation. Indeed, the branch related to  $m = 0$  grows after  $\Omega/f_{00} = 0.25$ , while in contrast, for  $m = 2$ , the frequency decreases after an initial increase. Regarding the torsional frequency, the curve monotonically decreases with the speed, as also shown in (Guo et al., 2001).

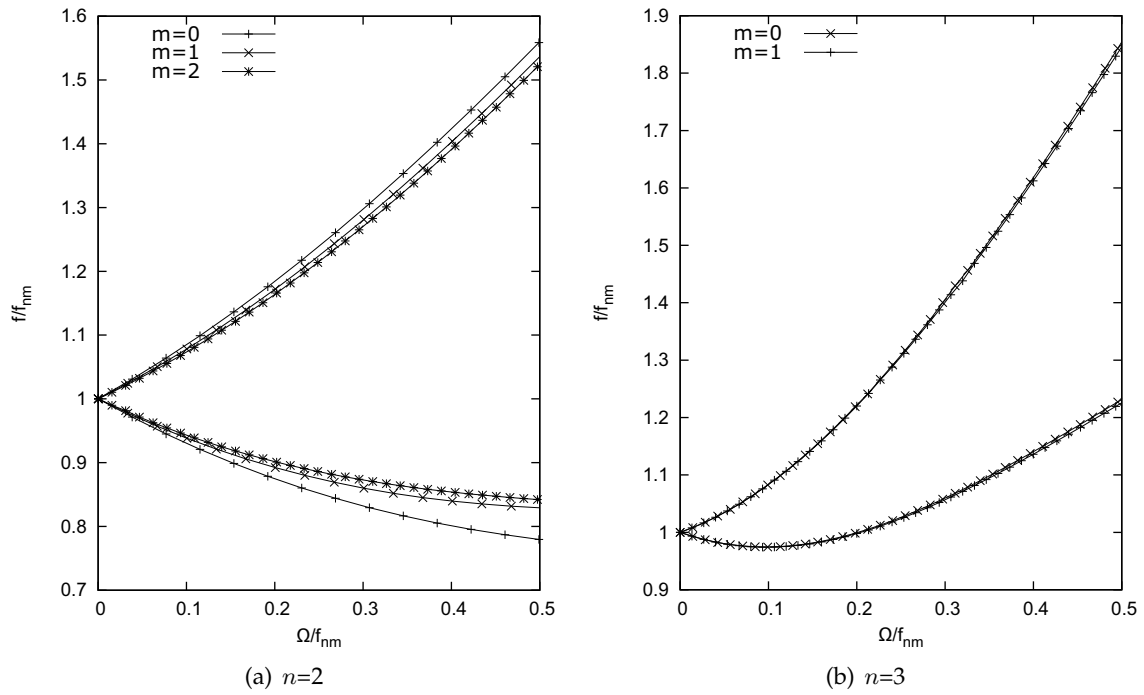
Similarly to the case of thin cylinders, Tab.5.25 lists forward and backward dimensionless frequencies for various speed parameters, with and without the initial stress. The structure is clamped at both ends, and the frequencies at standstill are  $f_2 = 4993.53$  Hz and  $f_3 = 8435.53$  Hz. Despite the thick wall, natural frequencies are considerably lower when the contribution of the initial stress is overlooked, especially for higher speed values. In addition, relative errors are very close to those computed for the thin cylinder in Tab.5.24.

### Laminated moderately thick cylinders

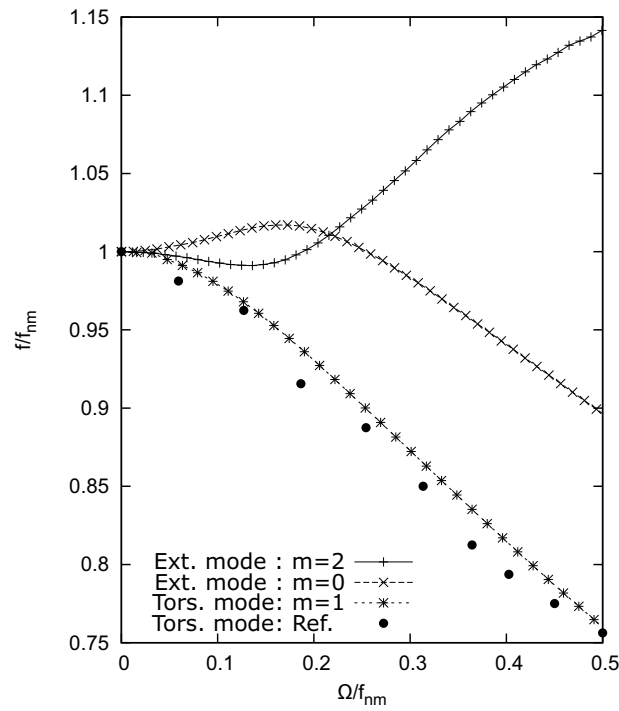
The effects of the rotational speed on the natural frequencies have been evaluated also for the laminated cylinders of Sec.5.2.3. Figures 5.27-5.29 show the frequency parameters as functions of the speed for various lamination schemes. In particular, in the first two graphs, bending fre-



**Figure 5.24** Dependency of frequency ratio upon the speed parameter of axial bending and global bending modes. TE9,  $n=1$ , Ref: (Guo et al., 2001).



**Figure 5.25** Dependency of frequency ratio upon the speed parameter of pure radial ( $m=0$ ) and radial shearing modes ( $m=1,2$ ). TE9.



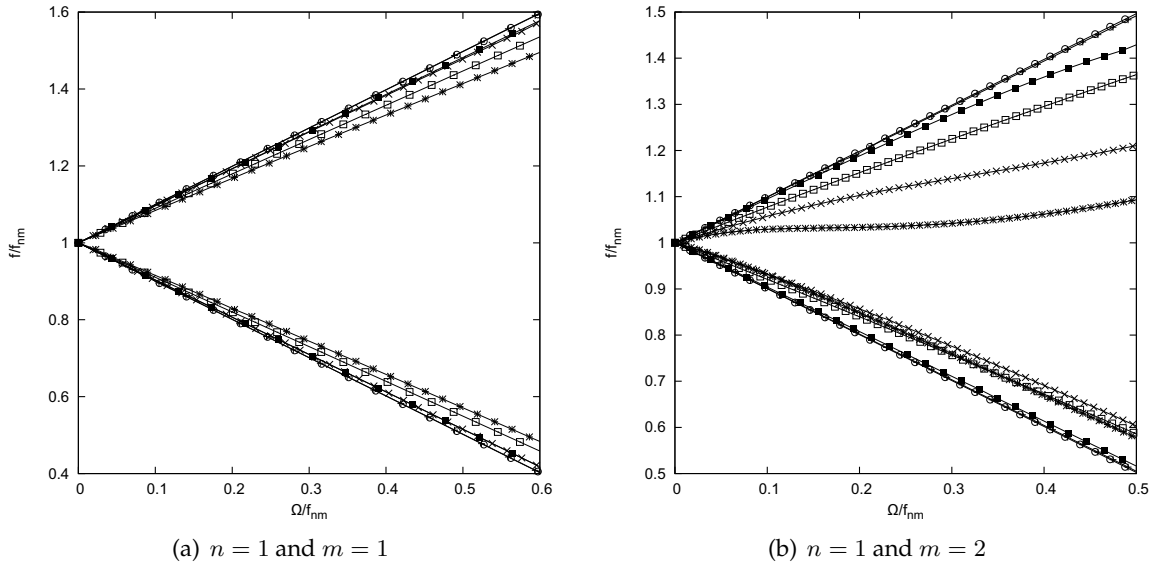
**Figure 5.26** Dependency of frequency ratio upon the speed parameter of extensional ( $n=0,2$ ,  $m=0$ ) and global torsion modes ( $n=0$ ,  $m=0$ ). TE9, Ref: (Guo et al., 2001).

|          |                 | $\Omega/f_n$         |                       |                        |                        |                        |                        |
|----------|-----------------|----------------------|-----------------------|------------------------|------------------------|------------------------|------------------------|
|          |                 | 0.0                  | 0.1                   | 0.2                    | 0.3                    | 0.4                    | 0.5                    |
| Forward  |                 |                      |                       |                        |                        |                        |                        |
| $f/f_2$  | with $\sigma_0$ | 1.0                  | 0.93                  | 0.88                   | 0.83                   | 0.80                   | 0.77                   |
|          | w/o $\sigma_0$  | 1.0 <sup>(0.0)</sup> | 0.92 <sup>(1.1)</sup> | 0.84 <sup>(4.5)</sup>  | 0.75 <sup>(9.6)</sup>  | 0.66 <sup>(17.5)</sup> | 0.56 <sup>(27.3)</sup> |
| $f/f_3$  | with $\sigma_0$ | 1.0                  | 0.97                  | 0.99                   | 1.05                   | 1.12                   | 1.2                    |
|          | w/o $\sigma_0$  | 1.0 <sup>(0.0)</sup> | 0.94 <sup>(3.1)</sup> | 0.87 <sup>(12.1)</sup> | 0.79 <sup>(24.8)</sup> | 0.71 <sup>(36.6)</sup> | 0.62 <sup>(48.3)</sup> |
| Backward |                 |                      |                       |                        |                        |                        |                        |
| $f/f_2$  | with $\sigma_0$ | 1.0                  | 1.09                  | 1.18                   | 1.29                   | 1.45                   | 1.56                   |
|          | w/o $\sigma_0$  | 1.0 <sup>(0.0)</sup> | 1.07 <sup>(1.8)</sup> | 1.14 <sup>(3.4)</sup>  | 1.20 <sup>(7.5)</sup>  | 1.25 <sup>(13.8)</sup> | 1.30 <sup>(16.7)</sup> |
| $f/f_3$  | with $\sigma_0$ | 1.0                  | 1.08                  | 1.22                   | 1.40                   | 1.62                   | 1.86                   |
|          | w/o $\sigma_0$  | 1.0 <sup>(0.0)</sup> | 1.05 <sup>(2.8)</sup> | 1.09 <sup>(10.6)</sup> | 1.12 <sup>(20.0)</sup> | 1.13 <sup>(30.2)</sup> | 1.14 <sup>(38.7)</sup> |

In brackets, relative errors between the two cases.

**Table 5.25** Dimensionless frequencies  $f/f_n$  vs. dimensionless speed  $\Omega/f_n$  with and without the hoop tension for the thick cylinder.

quencies vary almost linearly only when the lamination schemes are  $[0^\circ/0^\circ]$  and  $[0^\circ/90^\circ/0^\circ]$ . The non-linear trend of the other curves is much more evident for the angle ply configurations and when the longitudinal wave number,  $m$ , is equal to 2 (Fig.5.27-b). Figures 5.28-a and 5.28-b show the forward and backward branches for  $n$  equal to 2 and 3, respectively. Although the influence of the stacking sequence decreases when ' $n$ ' increases, there remain differences in the frequency variations at higher speed values. Finally, Fig.5.29 reveals that the torsional branch variations also depend on the lamination scheme.



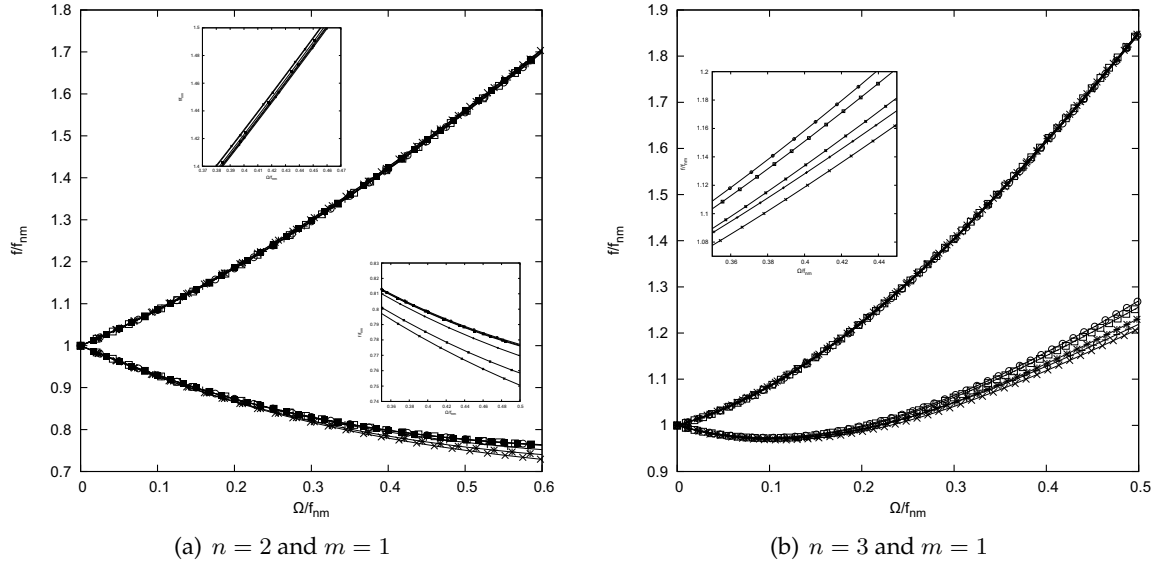
**Figure 5.27** Dependency of the frequency ratios upon the speed parameter for two different longitudinal wave number. TE8. '-+-':  $[0^\circ/0^\circ]$ , '-x-':  $[30^\circ/30^\circ]$ , '-\*-':  $[45^\circ/45^\circ]$ , '-□-':  $[65^\circ/65^\circ]$ , '-■-':  $[90^\circ/90^\circ]$ , '-o-':  $[0^\circ/90^\circ/0^\circ]$ .

## Rotating ring

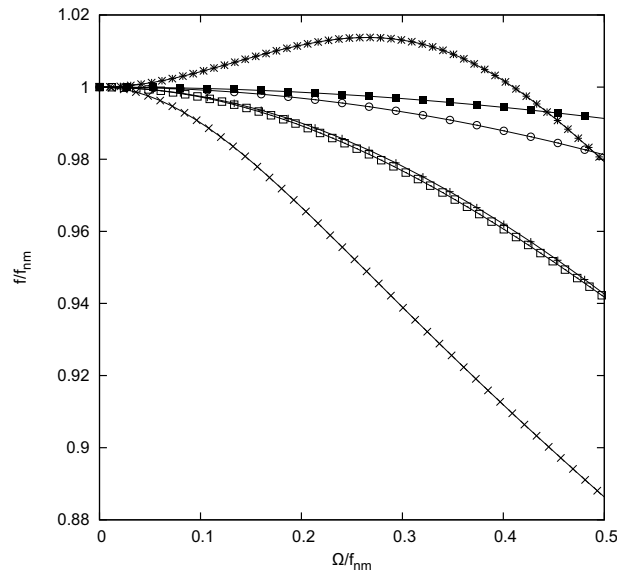
We now consider a thin ring, rotating about its centre. The mean radius is 0.5 m, and the rectangular cross-section has the width and the height equal to 0.05 and 0.02 m, respectively. The structure made of aluminium (Young's Modulus,  $E=73.1$  GPa; density,  $\rho=2770$  kg/m<sup>3</sup>; Poisson's ratio,  $\nu=0.33$ ) has been modelled using 40 9-node Lagrange elements above the cross-section, and 4 2-node beam elements along the longitudinal direction (the total number of degrees of freedom is 3600). Figure 5.30 shows the first natural frequencies as a function of the spin speed and the results are compared with those presented in (Genta, 2005), which were obtained with 40 brick 20-node elements. It should be noted that the 1D results strongly agree with the reference solution for all frequencies, which correspond to out-of-plane and in-plane mode shapes.

## Rotating disc

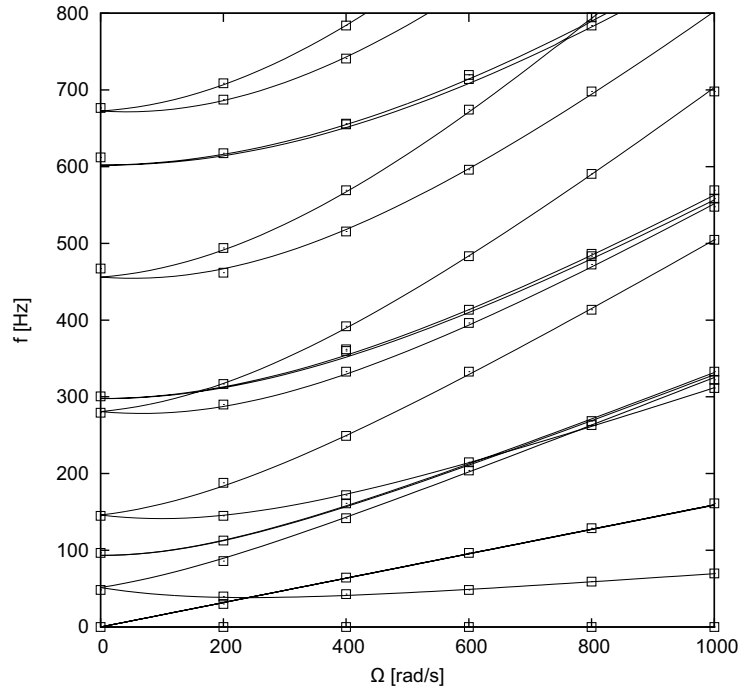
Since the Lagrange-type beam elements allows one to impose non-classical boundary conditions, a thin hollow disc, clamped at the bore, is now considered. The outer radius, the inner radius, and the uniform thickness are 203.2, 101.6 and 1.016 mm, respectively. The material properties are  $E=2.1$  GPa,  $\rho=7800$  kg/m<sup>3</sup>, and  $\nu=0.3$ . The cross-section has been mod-



**Figure 5.28** Dependency of the frequency ratios upon the speed parameter for two different circumferential wave number. TE8. '-+-':  $[0^\circ/0^\circ]$ , '-x-':  $[30^\circ/30^\circ]$ , '-\*-':  $[45^\circ/45^\circ]$ , '-□-':  $[65^\circ/65^\circ]$ , '-■-':  $[90^\circ/90^\circ]$ , '-o-':  $[0^\circ/90^\circ/0^\circ]$ .



**Figure 5.29** Dependency of the frequency ratios upon the speed parameter for the torsional mode. TE8,  $n = 1$ ,  $m = 0$ . '-+-':  $[0^\circ/0^\circ]$ , '-x-':  $[30^\circ/30^\circ]$ , '-\*-':  $[45^\circ/45^\circ]$ , '-□-':  $[65^\circ/65^\circ]$ , '-■-':  $[90^\circ/90^\circ]$ , '-o-':  $[0^\circ/90^\circ/0^\circ]$ .



**Figure 5.30** Frequencies [Hz] vs. spin speed [rad/s] for a rotating ring; '—': LE9, '□': solid FE model (Genta, 2005)

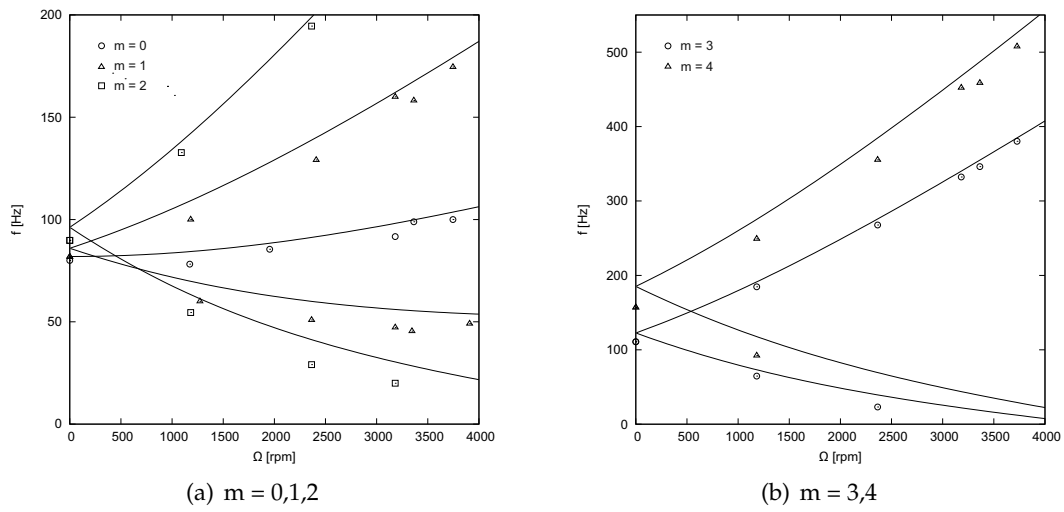
elled with 40 9-node Lagrange elements (4 in the radial direction, and 10 along the circumference) but, contrary to the previous case, a single 4-node beam element has been used along the longitudinal direction. The total number of degrees of freedom of this mathematical model is 2160. Figures 5.31 graphically show the evolutions of frequencies with the spin speed for modes characterised by a different number of nodal diameters, ' $m$ '. The frequencies have been reported in the inertial frame in order to compare the present solution with the experimental data reported in (Mote and Nieh, 1971). Despite slight differences in the computation of frequencies at  $\Omega=0$ , graphs reveal that the unified 1D formulation accurately describes the dynamics of highly deformable structures with a low number of degrees of freedom.

### Rotating disc on a flexible shaft

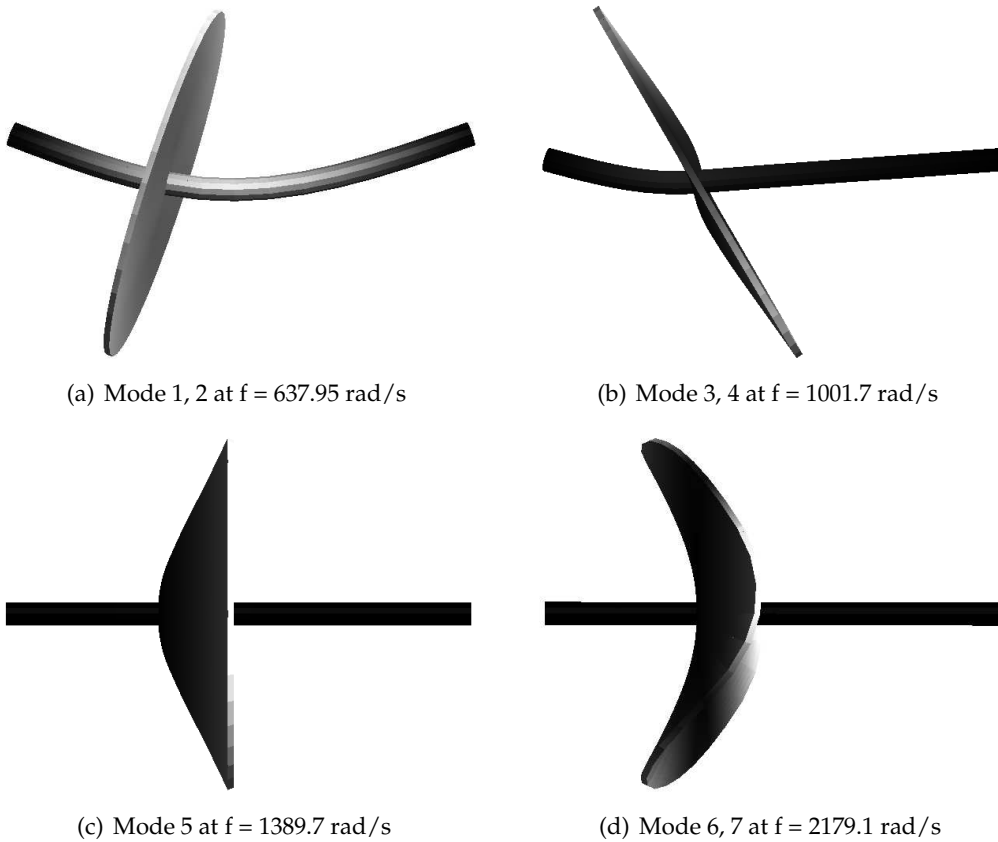
The rotor is constituted by a thin disc, whose radius and thickness are  $r_d=0.15$ ,  $h_d=0.005$  m, fixed at one-third of a shaft with length,  $L_s=0.4$  m, and radius,  $r_s=0.01$  m. The rotor is simply supported at each extremity and it is made of steel ( $E=207$  GPa,  $\rho=7860$  kg/m<sup>3</sup>, and  $\nu=0.28$ ). The used mathematical model (2010 degrees of freedom) consisted of 12 L3 for the shaft cross-section, 24 L9 for the disc, and 7 4-node beam elements along the longitudinal axis. Figure 5.32 shows the first 7 mode shapes at  $\Omega=0$ , computed with the proposed beam model.

It should be noted that, except the first 2 modes, the deformability of the disc strongly affects the dynamics of the whole structure. Moreover, in Fig.5.33, the corresponding forward and backward frequencies are compared with results presented in (Combesure and Lazarus, 2008), where Combesure *et al.* provided a solid finite element solution. The comparison reveals that the 1D-CUF formulation leads to accurate results with a limited computational effort demonstrating that complex-shaped rotors can be easily studied with the present approach.

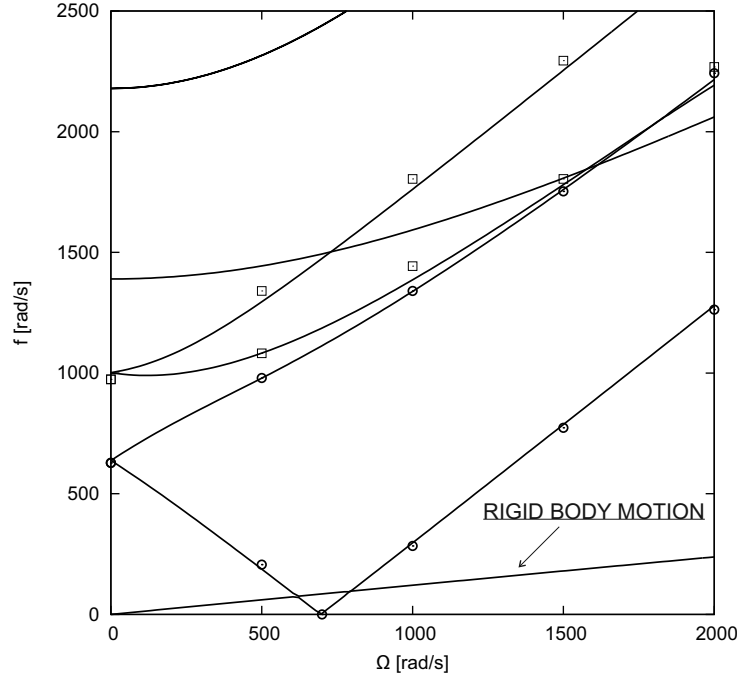




**Figure 5.31** Frequencies [Hz] vs. spin speed [rpm] for a rotating disc; '—': LE9, '○', '△', '□': experimental data (Mote and Nieh, 1971).



**Figure 5.32** First seven mode shapes of the disc on the flexible shaft in the non-rotating state.



**Figure 5.33** Frequencies vs. spin speed for a rotating disc on a flexible shaft; '—': LE model, 'o', '□': solid FE model (Combescure and Lazarus, 2008)

## 5.4 The centrifugally stiffened structures

To assess the theory of Sec.3.3, several illustrative examples are presented. The boundary conditions and the beam dimensions are the problem parameters. To enable a general application of results, if not otherwise declared, they are presented in non-dimensional form adopting the following expressions:

$$\omega^* = \frac{\omega}{\omega_0}, \quad \Omega^* = \frac{\Omega}{\omega_0}, \quad \omega_0 = \sqrt{\frac{\rho A L^4}{E J_{xx}}} \quad \delta_t = \frac{r_h}{L} \quad S = \sqrt{\frac{A L^2}{J_{xx}}}$$

where  $J_{xx}$  is the moment of inertia about x-axis,  $E$  is the Young Modulus and  $A$  the area of the cross-section. In the first test case, a thin rotating beam is considered and the non-dimensional results obtained in CUF form are compared with those presented in (Banerjee, 2000) where the Dynamic Stiffness Method were applied. As presented in Tab.5.26, the results strongly agree with the reference for all kinds of boundary conditions (B.C.) and for the considered hub offset ratios. The boundary conditions are clamped-free ( $C - F$ ), clamped-supported ( $C - P$ ) and supported-supported ( $P - P$ ). To clarify, the comparison becomes possible if both Coriolis and the Softening matrices are ignored. In the second analysis, the structure is thicker ( $S = 30$ ) to introduce the approximations of Euler-Bernoulli model. To do so, the first order shear deformation theory is needed. The variation of the fundamental non-dimensional frequency of a cantilever Timoshenko beam with the rotating speed is shown in Tab.5.27 and it is compared with (Lee and Kuo, 1993), without the use of any shear correction factor. The agreement between the two sets of results is evident, in fact, the relative error remains below 1%.

Since the unified formulation allows variations of the kinematic theories, higher-order mod-

| $B.C.$  | $\omega^*$ | $Theory$         | $\Omega^* = 1$ |                | $\Omega^* = 5$ |                |
|---------|------------|------------------|----------------|----------------|----------------|----------------|
|         |            |                  | $\delta_t = 0$ | $\delta_t = 1$ | $\delta_t = 2$ | $\delta_t = 3$ |
| $C - F$ | 1          | (Banerjee, 2000) | 3.6816         | 3.8888         | 10.862         | 12.483         |
|         |            | Present          | 3.6816         | 3.8895         | 10.866         | 12.488         |
|         | 2          | (Banerjee, 2000) | 22.181         | 22.375         | 32.764         | 35.827         |
|         |            | Present          | 22.178         | 22.375         | 32.773         | 35.840         |
|         | 3          | (Banerjee, 2000) | 61.842         | 62.043         | 73.984         | 77.935         |
|         |            | Present          | 61.836         | 62.032         | 73.986         | 77.939         |
| $C - P$ | 1          | (Banerjee, 2000) | 15.513         | 15.650         | 22.663         | 24.729         |
|         |            | Present          | 15.512         | 15.649         | 22.670         | 24.738         |
|         | 2          | (Banerjee, 2000) | 50.093         | 50.277         | 60.906         | 64.382         |
|         |            | Present          | 50.092         | 50.275         | 60.919         | 64.399         |
|         | 3          | (Banerjee, 2000) | 104.39         | 104.59         | 116.99         | 121.30         |
|         |            | Present          | 104.42         | 104.62         | 117.04         | 121.36         |
| $P - P$ | 1          | (Banerjee, 2000) | 10.022         | 10.264         | 19.684         | 22.078         |
|         |            | Present          | 10.021         | 10.264         | 19.690         | 22.086         |
|         | 2          | (Banerjee, 2000) | 39.642         | 39.889         | 53.132         | 57.235         |
|         |            | Present          | 39.638         | 39.886         | 53.141         | 57.248         |
|         | 3          | (Banerjee, 2000) | 88.991         | 89.241         | 103.92         | 108.93         |
|         |            | Present          | 89.003         | 89.253         | 103.94         | 108.96         |

**Table 5.26** Dependency of the first three dimensionless natural frequency parameters on the variations of the dimensionless angular speed and hub dimension of a Euler-Bernoulli beam.

| $\Omega^*$ | (Lee and Kuo, 1993) | Present | $Diff.(\%)$ |
|------------|---------------------|---------|-------------|
| 0          | 3.4798              | 3.4831  | 0.09        |
| 1          | 3.6452              | 3.6494  | 0.11        |
| 2          | 4.0994              | 4.1064  | 0.17        |
| 3          | 4.7558              | 4.7667  | 0.22        |
| 4          | 5.5375              | 5.5530  | 0.27        |
| 5          | 6.3934              | 6.4144  | 0.32        |
| 6          | 7.2929              | 7.3205  | 0.37        |
| 7          | 8.2184              | 8.2538  | 0.43        |
| 8          | 9.1596              | 9.2039  | 0.48        |
| 9          | 10.109              | 10.164  | 0.54        |

**Table 5.27** Non-dimensional fundamental frequency  $\omega^*$  of a cantilever Timoshenko beam as a function of the rotating speed.

els are tested. The considered structures are two graphite-epoxy rectangular beams. For both beams, the width, the length and the hub off-set are 25.4, 800.01 and 63.5 mm, respectively. The thickness and the fiber orientations are different for the two cases and they are listed in Tab.5.28 with the material properties. The reference solutions related to the flapwise normal modes are taken from (Hodges et al., 1996), where some experimental data were reported in graphical form (Epps and Chandra, 1996). As shown in Fig.5.34, the frequencies obtained with CUF are in good agreement with the theoretical approaches and experimental data. It is interesting to note that, the second-degree polynomial is sufficient to detect the bending modes and their related frequency values whereas, for vibration modes dominated by torsion, further refinements are needed to improve the solutions. In fact, using the fourth-degree polynomial, the torsional mode frequencies decrease, getting close to reference values. The remaining frequencies labeled with  $B_C$  are related to chordwise modes.

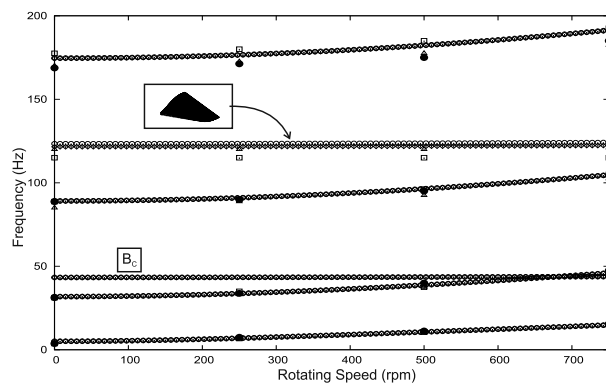
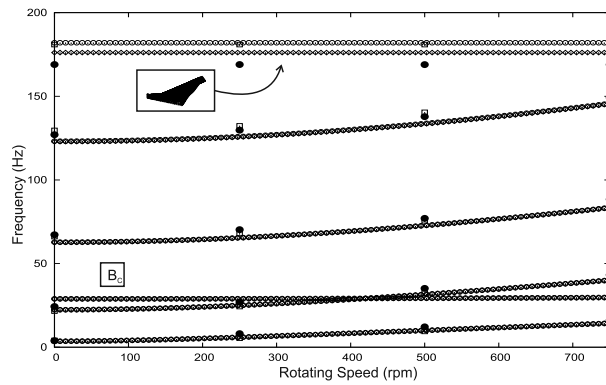
| <i>Case</i>           | <i>Thickness [mm]</i>  | <i>Orientation</i> |
|-----------------------|------------------------|--------------------|
| 1                     | 2.97                   | $0^\circ$          |
| 2                     | 3.22                   | $15^\circ$         |
| <i>Property</i>       | <i>Dimension [GPa]</i> |                    |
| $E_{11}$              | 141.9                  |                    |
| $E_{22} = E_{33}$     | 9.780                  |                    |
| $G_{21} = G_{31}$     | 6.140                  |                    |
| $G_{32}$              | 5.520                  |                    |
| $\nu_{12} = \nu_{13}$ | 0.42                   |                    |
| $\nu_{23}$            | 0.54                   |                    |

**Table 5.28** Material and dimensions of graphite-epoxy beams.

In order to investigate the effects of the ply angle and the rotational speed on the frequencies of more complex and realistic structures, we analyse a thin-walled box with the following geometry features and material properties

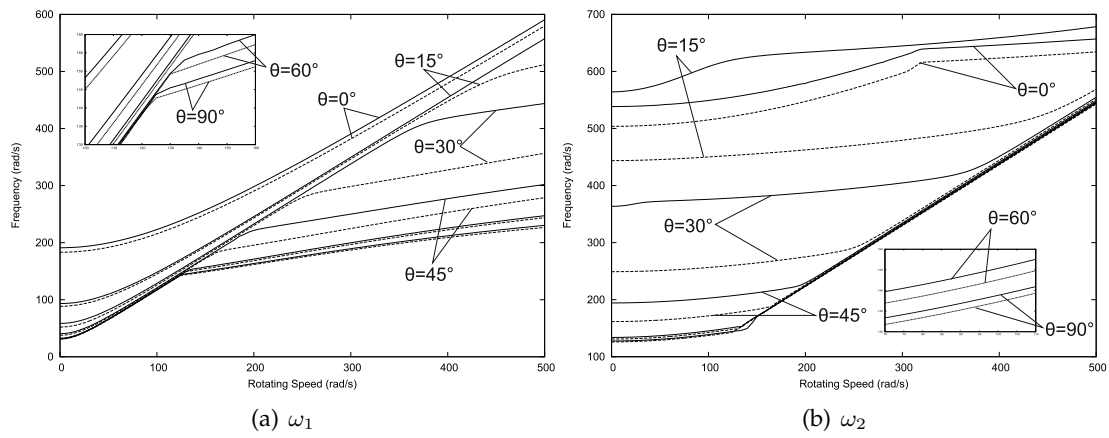
$$\begin{aligned}
 E_L &= 206.8 \text{ GPa} & E_T &= 5.17 \text{ GPa} & G_{LT} &= 2.55 \text{ GPa} & G_{TT} &= 3.10 \text{ GPa} \\
 \nu_{LT} &= \nu_{TT} = 0.25, & \rho &= 1528.15 \text{ Kg/m}^3 \\
 b &= 0.0508 \text{ m} & c &= 0.2540 \text{ m} & t &= 0.01016 \text{ m}
 \end{aligned}$$

where  $L$  and  $T$  denote parallel and transverse directions to the fibers,  $b$  is the width and  $t$  the thickness of the beam wall. The beam and hub lengths are assumed to be 2.032 m. In (Song and Librescu, 1997b), the dependency of the first angular frequency on both rotational speed and ply angle orientations was evaluated using a refined theory, where some non-classical effects such as torsion were taken into account. However, in that case, Coriolis terms were discarded in order to split the equations describing the flap-lag bending deformation from those related to the extension-twist motion. In our test case, Coriolis matrix and the term  $u_{y,y}$  are included. Figure 5.35 shows the first two angular frequencies changing with the rotational speed for different fiber orientations. The solid lines are obtained with the second-order expansion TE2, whereas the dotted curves with the TE4 expansion. With these two displacement models, it

(a)  $0^\circ$ (b)  $15^\circ$ 

**Figure 5.34** Frequencies versus angular speed - "o": Present (TE2), "◇": Present (TE4), "□": Theory (Hodges et al., 1996), "△": Theory (Epps and Chandra, 1996) and "●": Experimental data (Epps and Chandra, 1996).

is possible to detect the flexural-twist coupling when the ply angles are different from  $0^\circ$  and  $90^\circ$ . Focusing on Fig.5.35-a, under a certain value of rotational speed depending on the fiber orientation, the dominant modal shape that exhibits the lowest frequency is the flapwise mode. Due to the centrifugal effect, the frequency rapidly increases until the lag mode frequency becomes the lower of the two. The same phenomenon is observed in Fig.5.35-b for the second angular frequency but, in this case, the threshold speeds determine the interchange between the lag/torsional modal shapes and those that involve flap/extensional deformations. Moreover, for both cases, the curves computed with TE4 undergo a translation toward lower values, whereas the trends remain almost similar.



**Figure 5.35** First angular frequency versus rotational speed for different ply angles - "solid lines": TE2; "dotted lines": TE4

## 5.5 Flutter Analyses

### 5.5.1 Wing structures

A cantilevered straight plate is considered in the first numerical example. The length ( $L$ ), the thickness ( $t$ ) and the width ( $c$ ) of the investigated model are 0.305, 0.0762 and 0.001 m, respectively. The material, whose properties are  $E = 73.8$  GPa,  $\nu = 0.3$  and  $\rho = 2768$  Kg/m<sup>3</sup>, is isotropic. Both flutter velocity [m/s] and frequency [Hz] are computed using both steady (ST) and unsteady (TM) models with different displacement expansions. These results are reported in Tab.5.29 and compared with those found in (Petrolo, 2011), where the Doublet Lattice Method (DLM) was combined with Carrera's Unified Formulation. In this case, the use of the second-order Taylor-like expansion provides acceptable results with a very low number of degrees of freedom. It should be noted that, with the corrected lift coefficient, the flutter velocities ( $V_F$ ) computed with the TM are very close to the reference values while, with the ST, they are generally lower. However, the steady theory provided higher values of the flutter frequencies, especially with respect to those computed using Theodorsen's model.

**Table 5.29** Flutter conditions for an isotropic plate.

|     | ST     |          | TM     |          | DLM (Petrolo, 2011) |        |
|-----|--------|----------|--------|----------|---------------------|--------|
|     | $V_F$  | $f_F$    | $V_F$  | $f_F$    | $V_F$               | $f_F$  |
| TE2 | (50.2) | (45.000) | (60.6) | (36.823) | -                   | -      |
|     | 56.1   | 45.435   | 67.6   | 35.945   | 69.4                | 40.002 |
| TE3 | (49.5) | (42.904) | (59.5) | (35.257) | -                   | -      |
|     | 55.6   | 43.622   | 66.2   | 34.363   | 68.5                | 39.029 |
| TE4 | (49.8) | (43.395) | (59.5) | (35.244) | -                   | -      |
|     | 55.6   | 43.607   | 66.2   | 34.350   | 68.4                | 38.995 |

( ): the lift coefficient is not corrected.

The effects of the sweep angle ( $\Lambda$ ) are then investigated for the same structure. The ST and TM results are compared in Tab.5.30 with those presented in (Pagani et al., 2014), where the Dynamic Stiffness Method (DSM) was combined with DLM in CUF framework. The analyses of swept configurations require the use of more refined displacement theories in order to describe the bending-torsional coupling. For this reason, the third-order theory (TE3) is necessary to ensure convergence of the results. A positive angle of sweep implies a decrease in the flutter conditions. The steady theory again predicts lower flutter velocities and higher frequencies than the TM and DLM, which yield similar velocity values.

Other analyses are carried out on straight and swept composite plates. A six-layer cantilevered beam is first considered, using the same length as in previous structures, and total thickness and width 0.804 and 76.2 mm, respectively. The properties of the orthotropic material are  $E_L = 98$  GPa,  $E_T = 7.90$  GPa,  $G_{LT} = 5.60$  GPa,  $\nu = 0.28$  and  $\rho = 1520$  kg/m<sup>3</sup>. The flutter conditions for the stacking sequence  $[30_2/0]_S$  are compared in Tab.5.31 with those predicted by the DLM (Petrolo, 2013), using three different displacement theories. Despite the discrepancies between the flutter frequencies, the three aerodynamic approaches again predict similar flutter velocities for this case. This is also confirmed in Tab.5.32 where, for different lamination schemes, the flutter velocities computed using the TE3 displacement model are compared with 2D (CLT) and experimental (EXP) solutions.

**Table 5.30** Flutter conditions for an isotropic plate.

| $\Lambda$  | Theory                   | $V_F$ [m/s] |        |        | $f_F$ [Hz] |        |        |
|------------|--------------------------|-------------|--------|--------|------------|--------|--------|
|            |                          | TE2         | TE3    | TE4    | TE2        | TE3    | TE4    |
| $0^\circ$  | ST                       | 56.086      | 55.595 | 55.589 | 45.435     | 43.607 | 43.622 |
|            | TM                       | 67.564      | 66.247 | 66.231 | 35.945     | 34.363 | 34.350 |
|            | DLM(Pagani et al., 2014) | 69.388      | 68.503 | 68.523 | 40.002     | 39.029 | 38.995 |
| $10^\circ$ | ST                       | 55.949      | 54.630 | 54.609 | 41.602     | 40.774 | 40.708 |
|            | TM                       | 64.270      | 63.019 | 62.986 | 32.912     | 32.193 | 32.155 |
|            | DLM(Pagani et al., 2014) | 65.441      | 64.305 | 64.506 | 38.362     | 37.361 | 37.352 |
| $20^\circ$ | ST                       | 58.665      | 53.853 | 53.721 | 38.914     | 37.791 | 37.635 |
|            | TM                       | 64.845      | 60.352 | 60.210 | 31.223     | 29.859 | 29.770 |
|            | DLM(Pagani et al., 2014) | 66.408      | 61.046 | 59.130 | 36.736     | 35.095 | 34.793 |
| $30^\circ$ | ST                       | 63.627      | 53.147 | 52.714 | 35.803     | 33.977 | 33.750 |
|            | TM                       | 68.397      | 57.938 | 57.514 | 29.885     | 27.108 | 26.864 |
|            | DLM(Pagani et al., 2014) | 70.145      | 57.747 | 57.216 | 34.156     | 31.887 | 31.616 |

**Table 5.31** Flutter condition for a six-layer plate.  $[30_2/0]_S$ 

|     | ST     |        | TM     |        | DLM (Petrolo, 2013) |        |
|-----|--------|--------|--------|--------|---------------------|--------|
|     | $V_F$  | $f_F$  | $V_F$  | $f_F$  | $V_F$               | $f_F$  |
| TE2 | 25.969 | 27.185 | 28.726 | 23.132 | 28.820              | 27.813 |
| TE3 | 24.007 | 26.318 | 26.696 | 22.142 | 25.938              | 26.651 |
| TE4 | 23.915 | 26.216 | 26.590 | 22.038 | 25.937              | 26.650 |

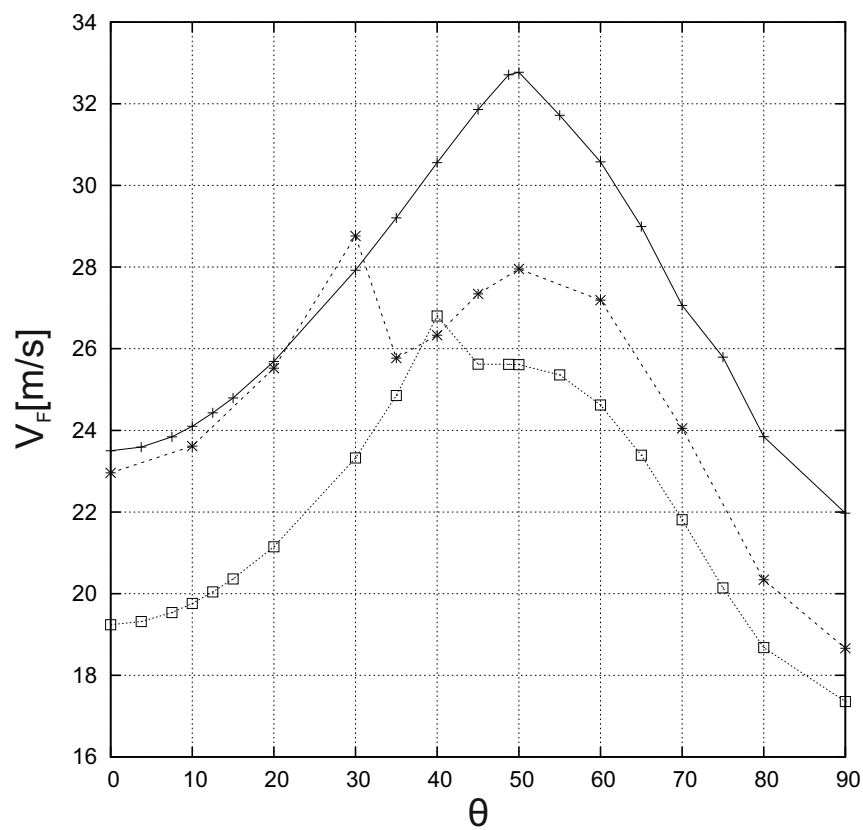
The influence of the lamination scheme on the flutter conditions is also evaluated considering the  $[\theta_2/90_2^\circ/-\theta_2]$  configuration. Figure 5.36 shows the instability velocities computed with the TE3 displacement model as a function of the angle-ply  $\theta$ . Even though the steady approach yielded more conservative results, it should be noted that the strip theories predict similar variations of the flutter speeds, which reach maximum values between of  $40^\circ$  and  $50^\circ$ . Instead, when the DLM is adopted, the flutter speed initially increases with the angle, and then rapidly decreases when  $\theta$  is equal to  $35^\circ$ . After this drop, the DLM results fall between those of the two strip theories. Figure 5.37 also shows both the frequencies and velocities of flutter computed with the steady and unsteady strip theories for the symmetric lamination schemes  $[\theta/\theta/0^\circ/0^\circ]_S$  as a function of the lamination angle. Again the theories qualitatively yield the same variations for these configurations, which involve bending-torsional coupling, and predict maximum values of the velocities and frequencies for  $150^\circ < \theta < 160^\circ$ .

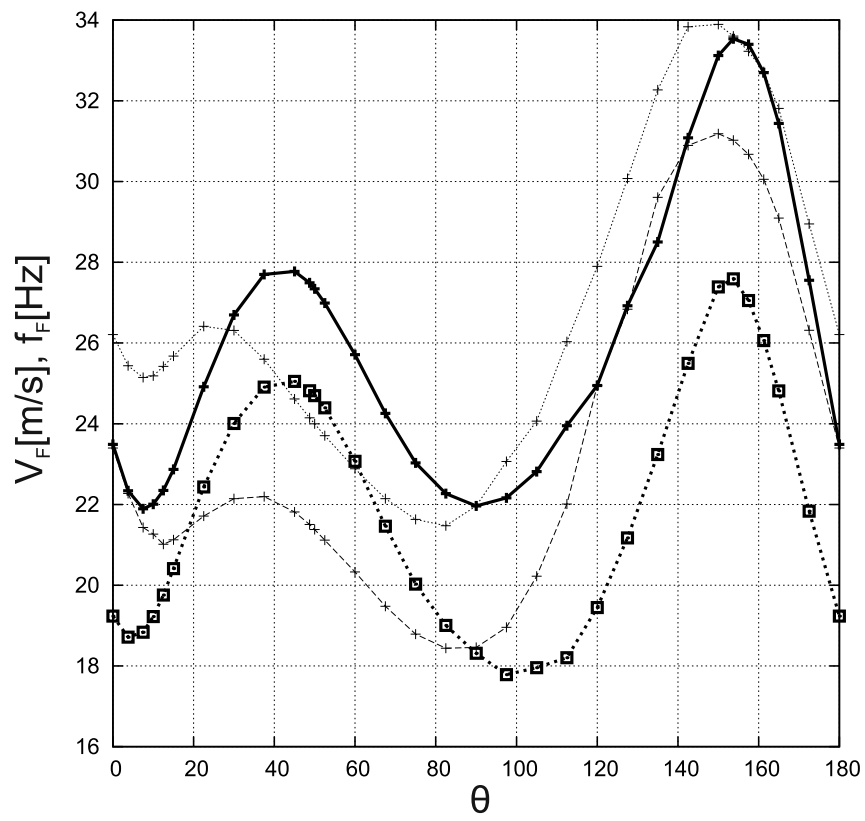
A symmetric eight-layer lamination is adopted for the beam configuration. The total thickness of the laminate is equal to 0.804 mm; the stacking sequence is  $[-22.5/67.5/22.5/-67.5]_S$ , and the thicknesses of plies are 0.03678, 0.04824, 0.06432 and 0.25326 mm, respectively. The natural



**Table 5.32** Flutter velocities for a six-layer plate, TE3.

| Lamination     | ST   | TM   | CLT(2007) | EXP.(1984) | DLM(Petrolo, 2013) |
|----------------|------|------|-----------|------------|--------------------|
| $[0_2/90]_S$   | 19.2 | 23.5 | 23.0      | 25         | 23.2               |
| $[45/-45/0]_S$ | 40.1 | 44.2 | 40.1      | >32        | 40.3               |
| $[45_2/0]_S$   | 25.0 | 27.8 | 27.5      | 28         | 26.2               |
| $[30_2/0]_S$   | 24.0 | 26.7 | 27.1      | 27         | 25.9               |

**Figure 5.36** Flutter velocities on the angle ply for the lamination case  $[\theta_2/90_2^o/-\theta_2]$ : — \* —: DLM, — + —: TM, — □ —: ST.



**Figure 5.37** Flutter velocities (bold lines) and frequencies (lines) on the angle ply for the lamination case  $[\theta/\theta/0^\circ]_S$ :  
 $- + -$ : TM,  $- \square -$ : ST.

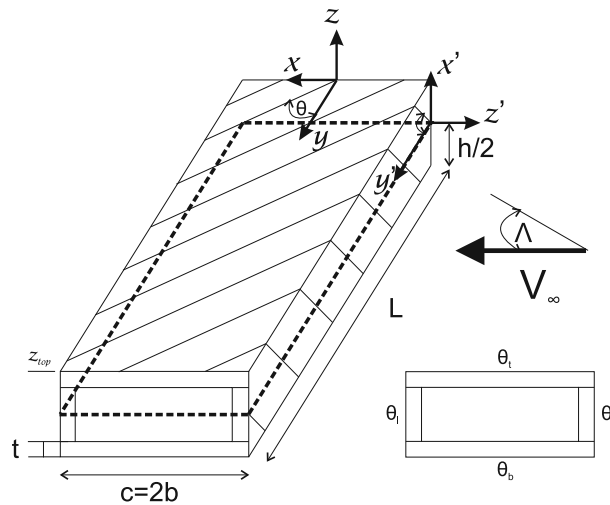
frequencies (bending and torsion), flutter conditions and the results from the CLT model are presented in Tab.5.33. The above considerations are also valid for this case.

**Table 5.33** Flutter velocities [m/s] for an eight-layer plate, TE3.

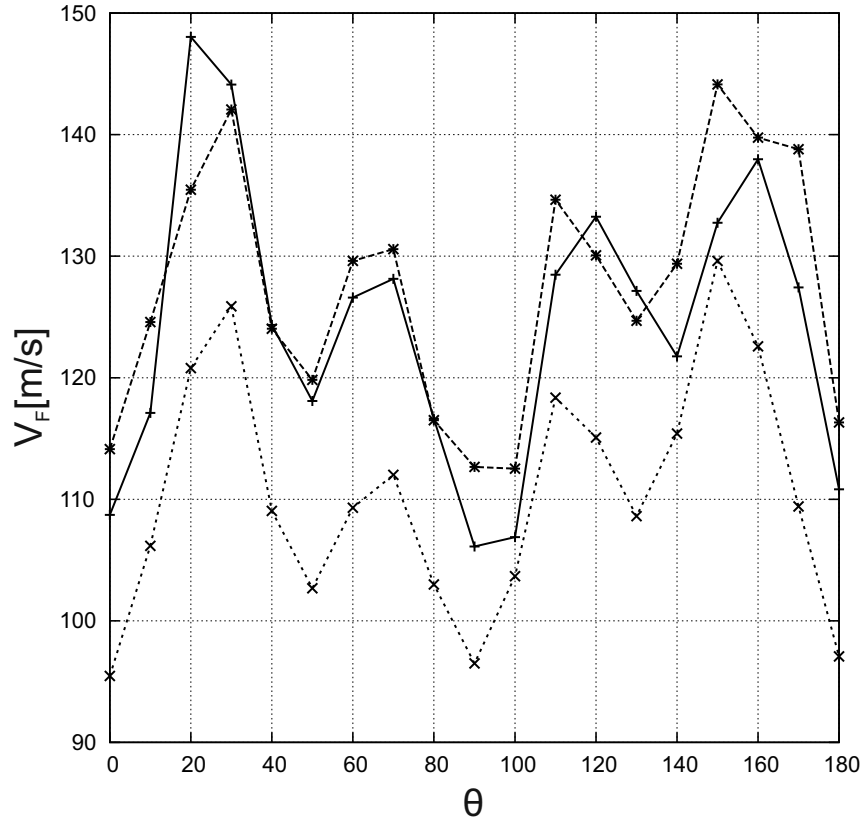
|                      | $f_1$ | $f_2$ | $f_3$ | $f_4$ | $f_5$ | $V_F$ |
|----------------------|-------|-------|-------|-------|-------|-------|
| $\Lambda = 0^\circ$  |       |       |       |       |       |       |
| ST                   | 7.2   | 45.1  | 59.1  | 126.5 | 182.5 | 30.7  |
| TM                   | 7.2   | 45.1  | 59.1  | 126.5 | 182.5 | 36.5  |
| DLM (Petrolo, 2013)  | 7.2   | 45.1  | 58.9  | 126.5 | 181.9 | 38.2  |
| CLT (2007)           | 7.3   | 45.4  | 59.1  | 127.7 | 182.3 | 38.8  |
| $\Lambda = 30^\circ$ |       |       |       |       |       |       |
| ST                   | 5.6   | 34.5  | 59.9  | 96.2  | 187.5 | 29.5  |
| TM                   | 5.6   | 34.5  | 59.9  | 96.2  | 187.5 | 32.9  |
| DLM (Petrolo, 2013)  | 5.6   | 34.4  | 59.6  | 95.9  | 182.7 | 32.0  |
| CLT (2007)           | 5.6   | 34.4  | 60.0  | 95.4  | 182.0 | 32.4  |

Following several studies, where the wing was modelled as a box structure, a prismatic thin-walled beam is considered. The structure, whose dimensions and material properties are listed in Tab.5.34, is discretized with seven 4-node beam-elements .

According to the  $[\theta_b/\theta_r/\theta_t/\theta_l]$  configuration of Fig.5.38 and, in order to consider an arbitrary lamination sequence where bending-torsion couplings are likely to occur, the  $[\theta/0.1\theta/2\theta/10\theta]$  case is studied. Adopting the fourth-order Taylor-like expansion (TE4), the flutter velocities are computed with the ST, TM and DLM approaches and their trends, as a function of the angle ply, are shown in Fig.5.39. The accuracy of the unsteady strip theory (TM) with respect to the panel method is acceptable, since the maximum relative difference is about 10%, while the steady aerodynamic theory always yielded lower speed values. Moreover, Tab.5.35 shows the flutter velocities for a number of angle values.



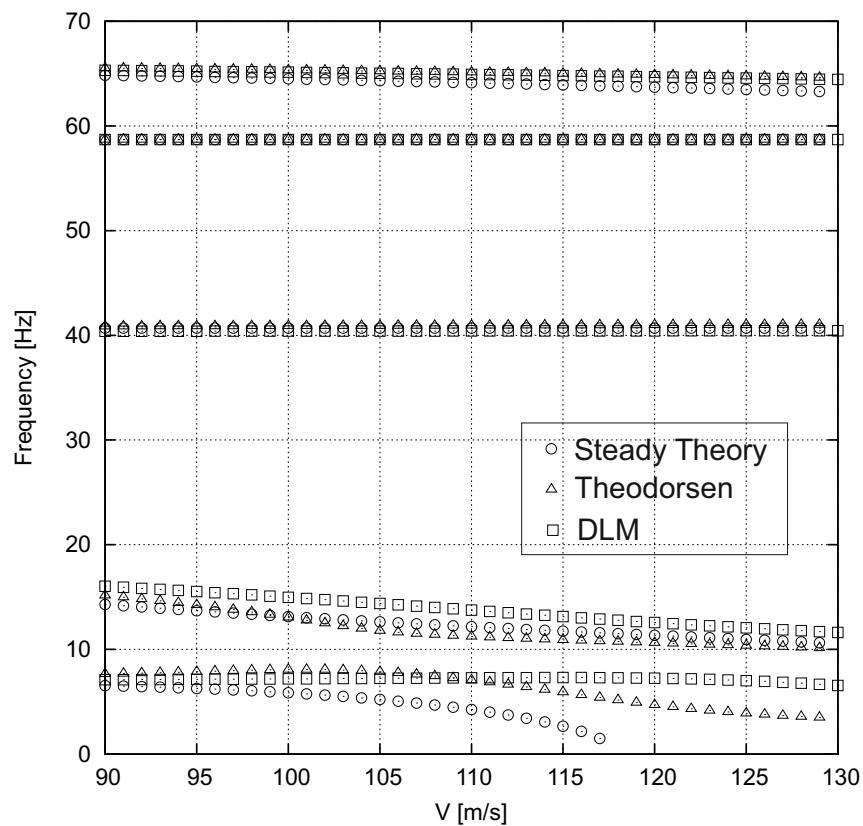
**Figure 5.38** Sketch of the box beam and coordinate reference system.



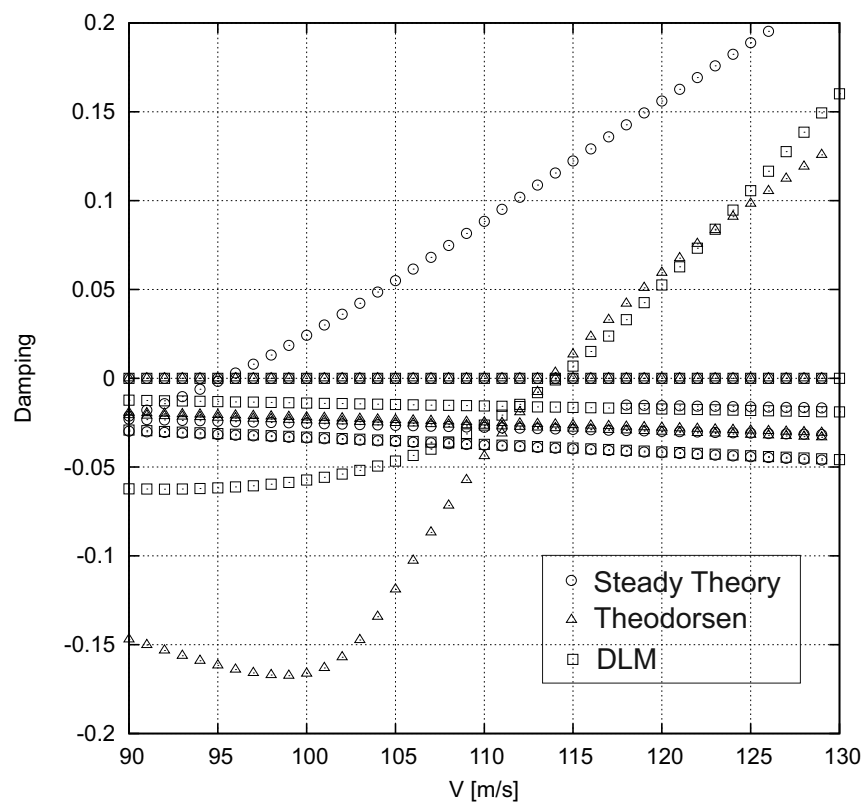
**Figure 5.39** Flutter velocities on the angle ply of the box beam:  $-\star-$ : DLM,  $-+-$ : TM,  $- \times -$ : ST.

Figure 5.40 shows the frequency and damping evolutions as functions of speed for  $\theta=0$  using the three aerodynamic approaches.

Although it was proved that the unsteady strip theory in the CUF framework provides good results, other analyses are carried out on the previous box structure, using the DLM, in order to compare different lamination solutions. The results shown in Fig.5.41 are normalized with respect to the flutter speed, when  $\theta$  is equal to  $0^\circ$ . As expected, the solution related to the first scheme  $[\theta/\theta/\theta/\theta]$  is symmetric with respect to the minimum flutter speed, which occurs for  $\theta$  equal to  $90^\circ$ . The two maximum speed values are reached for  $\theta$  equal to  $50^\circ$  and  $130^\circ$ . Although the worst condition is the same as that of the previous configuration, the second scheme  $([-\theta/-\theta/\theta/\theta])$  provides the highest flutter speed for the considered cases, when  $\theta$  is equal to  $20^\circ$ . Finally, regarding the last case  $[\theta/0.1\theta/2\theta/10\theta]$ , maximum value (at  $\theta = 150^\circ$ ) is lower than those of previous examples but, interestingly, the minimum speed is the highest (at  $\theta = 90^\circ$ ) with this configuration.

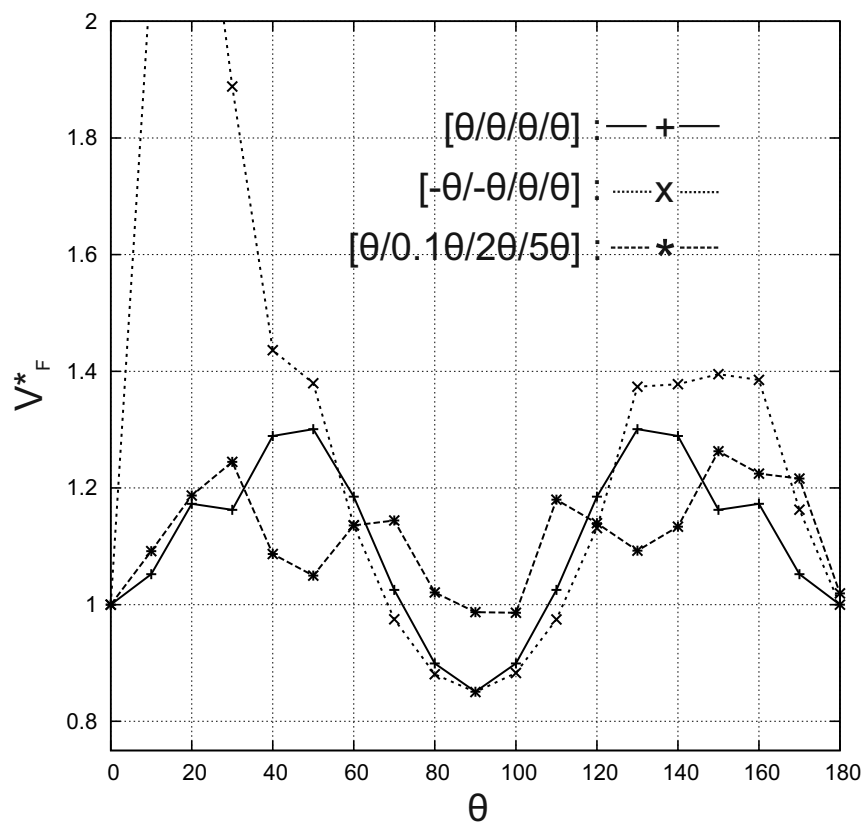


(a) Frequency



(b) Damping

**Figure 5.40** Frequency and damping values as functions of speed: TE4



**Figure 5.41** Normalized flutter velocities  $V_F^*$  on the angle ply of the box beam for three different lamination schemes (DLM).

**Table 5.34** Material properties and dimensions of the composite box beam

| <i>Property</i> | <i>Dimension</i>            |
|-----------------|-----------------------------|
| $E_{11}$        | 206.8 [Gpa]                 |
| $E_{22}$        | 5.17 [Gpa]                  |
| $G_{23}$        | 2.55 [Gpa]                  |
| $G_{31}$        | 2.55 [Gpa]                  |
| $G_{12}$        | 3.10 [Gpa]                  |
| $\nu_{12}$      | 0.25                        |
| $\rho$          | 1528.5 [kg/m <sup>3</sup> ] |
| $c$             | 0.5[m]                      |
| $h$             | c/15[m]                     |
| $t$             | c/150[m]                    |
| $L$             | 3.5[m]                      |

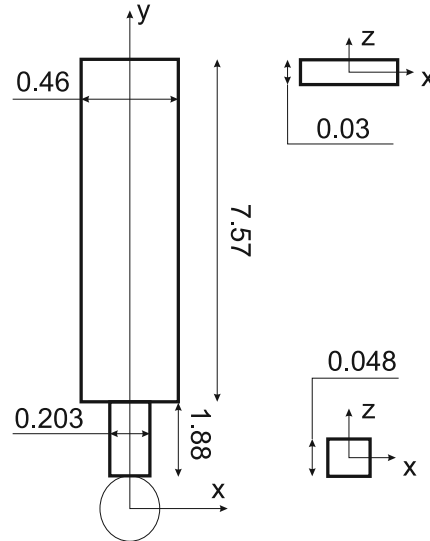
**Table 5.35** Flutter velocities [m/s] for the angle ply box beam, TE4.

|     | 0°     | 30°    | 60°    | 90°    | 120°   | 150°   | 180°   |
|-----|--------|--------|--------|--------|--------|--------|--------|
| ST  | 95.462 | 125.88 | 109.30 | 96.492 | 115.08 | 129.61 | 97.07  |
| TM  | 108.71 | 144.12 | 126.59 | 106.11 | 133.24 | 132.75 | 110.82 |
| DLM | 114.12 | 142.06 | 129.62 | 112.66 | 130.07 | 144.14 | 116.32 |

## 5.5.2 Centrifugally stiffened beams

### Isotropic rotating plate

The flutter conditions of an isotropic rotating plate are evaluated using different aerodynamic approaches. The material properties are  $E = 74$  GPa,  $\nu=0.33$  and  $\rho=2780$  kg/m<sup>3</sup> and, the geometrical features are shown in Fig.5.42.



**Figure 5.42** Geometrical features of the isotropic rotating plate.

**Table 5.36** Flutter velocities [rad/s] for the isotropic rotating plate, TE3.

|            | $\hat{h}$ | $f_F$ [Hz] | $\Omega_F$ [rad/s] |
|------------|-----------|------------|--------------------|
| Steady     | -         | 8.14       | 14.52              |
| Theodorsen | -         | 5.28       | 18.71              |
|            | 4.5       | 5.32       | 18.03              |
| Loewy      | 5.5       | 5.32       | 18.06              |
|            | 6.5       | 5.22       | 18.08              |

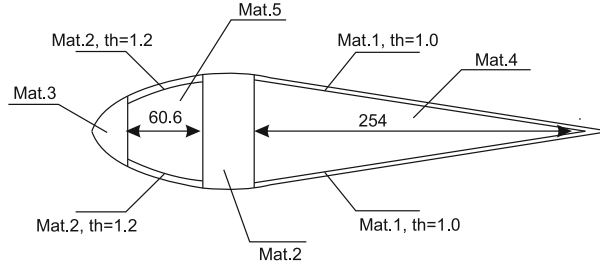
Table 5.36 shows the flutter conditions predicted using both steady and unsteady aerodynamic theories. As far as Loewy's model is concerned, three different spacing factors are considered in order to highlight its effects on the aeroelastic response of the blade. First of all, it is observed that the steady theory predicts a lower value of the flutter speed and a higher frequency with respect to the unsteady approaches. Secondly, despite different spacing factors, the unsteady models leads to very similar results both in terms of frequency and speed.

### Rotating multi-section blade: NACA 0012

To demonstrate the capabilities of the refined CUF beam elements, a realistic helicopter blade is here considered. The airfoil is a NACA 0012 and chord and length of the structure are 0.3454 and 5.186 m, respectively. Figure 5.43 shows the blade profile that is constituted by different sections with the materials listed in Tab.5.37. The structure is cantilevered and it is modelled



with 7 4-node beams. Several problem parameters are evaluated including the lamination angle of the skin ply, the aerodynamic theories and the order of TE models. Firstly, for the ply angle  $\theta$  equal to zero, Fig.5.44 shows the natural frequencies as a function of the rotational speed by adopting different displacement theories.



**Figure 5.43** Airfoil NACA 0012. The dimensions are in [mm].

**Table 5.37** Material properties of airfoil NACA 0012. Elastic moduli in [GPa] and densities in [kg/m<sup>3</sup>].

| Material | E <sub>11</sub> | E <sub>11</sub> | E <sub>11</sub> | G <sub>12</sub> | G <sub>31</sub> | G <sub>23</sub> | $\nu_{12}$ | $\nu_{31}$ | $\nu_{23}$ | $\rho$ |
|----------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|------------|------------|------------|--------|
| 1        | 139.0           | 11.0            | 11.0            | 6.05            | 6.05            | 3.78            | 0.313      | 0.313      | 0.313      | 1578   |
| 2        | 35.756          | 15.427          | 15.427          | 3.902           | 3.902           | 3.902           | 0.27       | 0.27       | 0.27       | 1826.9 |
| 3        | 1.446           | -               | -               | -               | -               | -               | 0.49       | -          | -          | 13683. |
| 4        | 0.0827          | -               | -               | -               | -               | -               | 0.2        | -          | -          | 190.99 |
| 5        | 0.03105         | -               | -               | -               | -               | -               | 0.45       | -          | -          | 95.48  |

—: the material is isotropic.

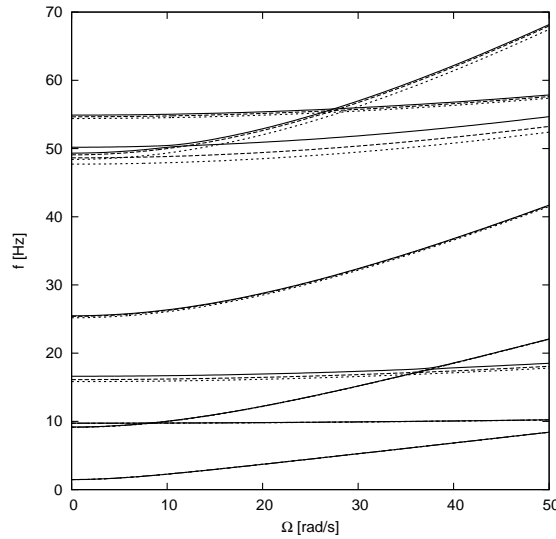
Despite slight differences in the computation of the torsional frequencies (4<sup>th</sup> and 6<sup>th</sup>), the TE3 model ensures acceptable accuracy with a very low number of degrees of freedom. On the other hand, since the choice of the stacking sequence can determine dramatic changes in the dynamic response of rotating structures, Fig.5.45 shows the frequency variations with the rotating speed for a number of lamination angles.

In Figs.5.44 and 5.45, it is possible to note at different speed values, points at which the frequency branches suddenly veer. The "veering" phenomenon determines a temporary coupling of the involved modal shapes that can, in unlucky cases, affect the stability of the rotating structure. In order to observe the evolutions of modes, the Modal Assurance Criterion (MAC) is here exploited. The MAC number is defined as a scalar representing the degree of consistency between two different modal vectors. The MAC takes on values from zero (representing no consistent correspondence) to one (representing a consistent correspondence) and it is defined as

$$\text{MAC}_{ij} = \frac{|\{\phi_{A_i}\}^T \{\phi_{A_j}\}|^2}{\{\phi_{A_i}\}^T \{\phi_{A_i}\} \{\phi_{A_j}\} \{\phi_{A_j}\}^T} \quad (5.10)$$

where  $\{\phi_{A_i}\}$  and  $\{\phi_{A_j}\}$  are the modal vector  $\{\phi_A\}$  at  $i^{\text{th}}$  and  $j^{\text{th}}$  speed. Figure 5.46 shows the MAC for speed values close to the "veering" point at approximately 35 [rad/s]. It is noteworthy that, after a phase where modal shapes differ from the original ones, they reappear simply interchanged as after a crossing. This changing can slowly happen increasing, therefore, the speed range where the deformation modes are coupled.

The flutter analyses are carried out by using the aerodynamic models proposed in Sec.4.2. Assuming that the four-blade rotor is operating at a thrust coefficient  $C_T = 0.014$ , the correspond-

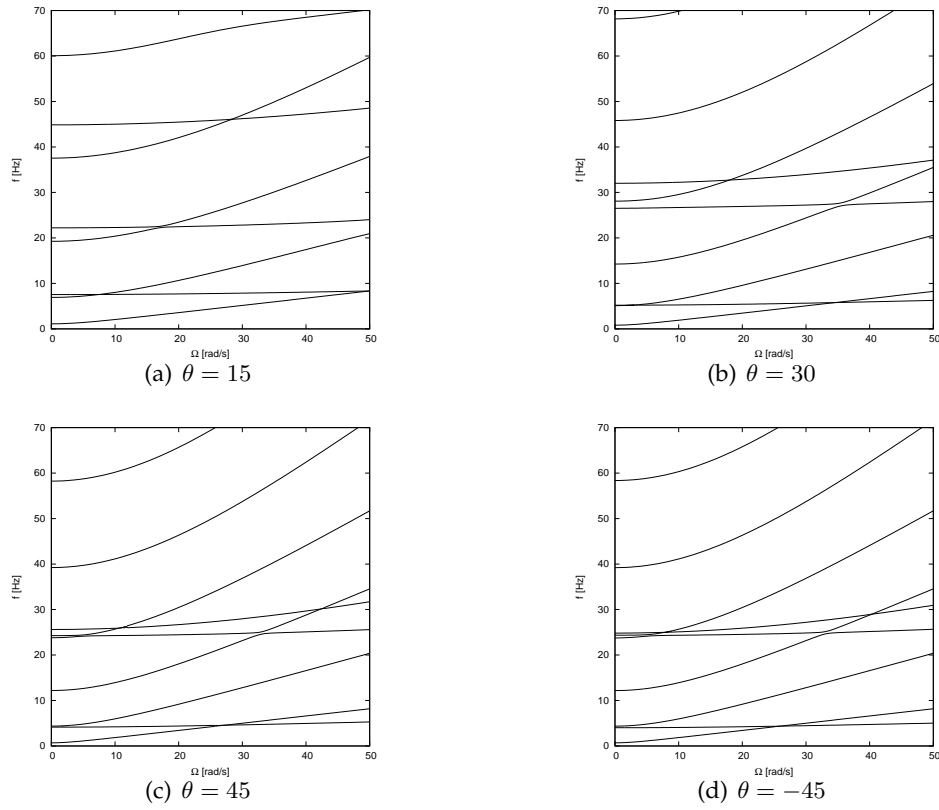


**Figure 5.44** Frequency vs rotational speed: '—' TE3, '---' TE4, '...' TE5.  $\theta = 0$

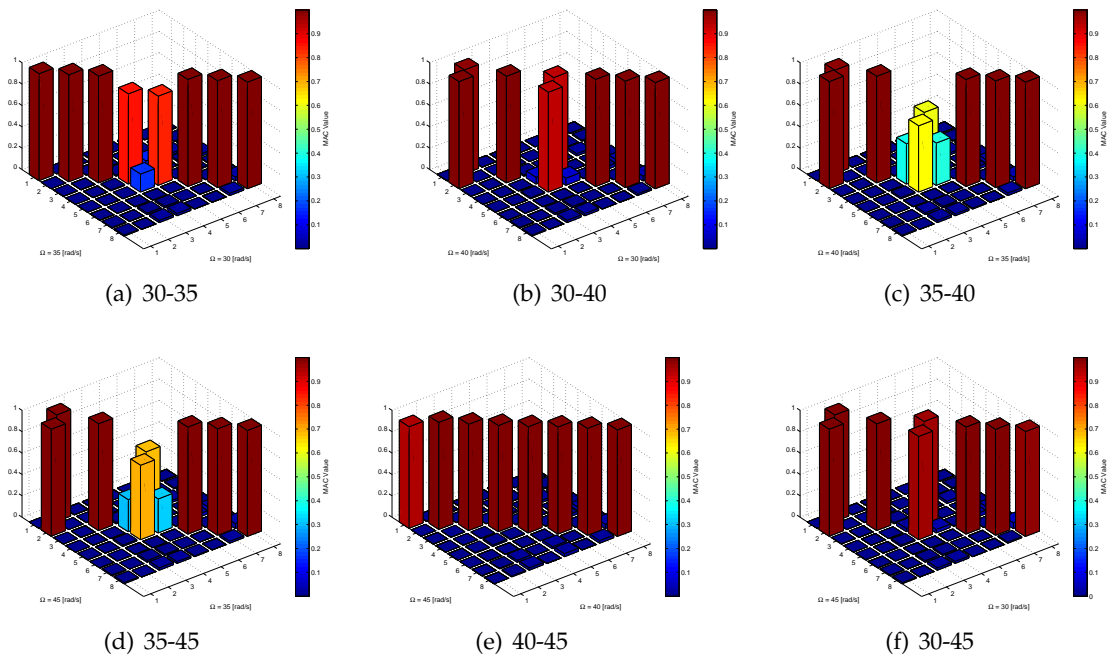
ing values of inflow ratio ' $\lambda_0$ ' and spacing factors ' $\hat{h}$ ' are 0.084 and 3. Table 5.38 shows the flutter conditions as a function of the lamination angle. The used displacement theories are TE3 and TE4 expansions. It is noteworthy that, in this case, the two unsteady theories lead to extremely different results when the angle is -15 and -30. According to Loewy's model, the lift deficiency function behaves as in Fig.5.47, where for the first reduced frequency shown in the box, the real part is almost equal to 1 while the imaginary is almost null. This fact means that the oscillatory lift excites the first deformation mode at each rotation causing, at a certain velocity, the instability. Clearly, this result is strongly affect by the choice of the spacing factor that represents, therefore, a crucial parameter.

**Table 5.38** Flutter velocities [rad/s] and frequencies [Hz] for the NACA-0012.

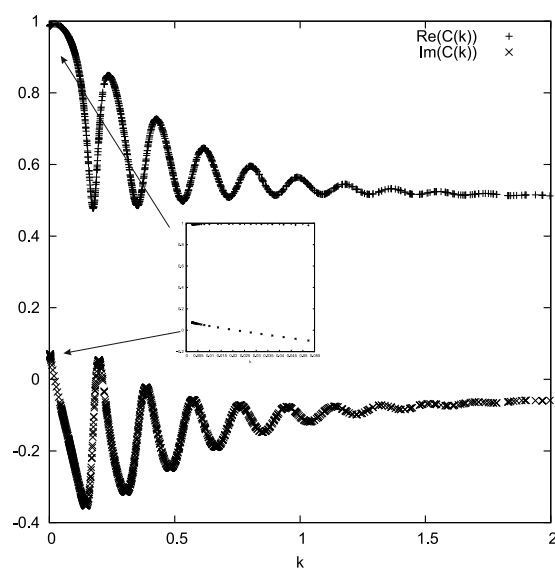
|            |     |       | $\theta$ |              |              |       |       |       |       |  |
|------------|-----|-------|----------|--------------|--------------|-------|-------|-------|-------|--|
|            |     |       | [-45]    | [-30]        | [-15]        | [0]   | [15]  | [30]  | [45]  |  |
| Steady     | TE3 | $V_F$ | 44.44    | 52.55        | 44.89        | 22.24 | 24.39 | 32.24 | 32.04 |  |
|            |     | $f_F$ | 16.03    | 16.15        | 13.10        | 13.50 | 17.76 | 21.05 | 19.14 |  |
|            | TE4 | $V_F$ | 39.64    | 49.24        | 42.92        | 22.25 | 23.48 | 30.64 | 30.32 |  |
|            |     | $f_F$ | 13.69    | 15.28        | 12.64        | 13.00 | 17.09 | 19.96 | 18.30 |  |
| Theodorsen | TE3 | $V_F$ | 46.72    | 54.31        | 47.72        | 30.28 | 27.47 | 36.40 | 36.80 |  |
|            |     | $f_F$ | 15.26    | 16.07        | 13.48        | 12.07 | 16.17 | 19.38 | 18.00 |  |
|            | TE4 | $V_F$ | 40.34    | 51.41        | 45.78        | 29.18 | 26.43 | 34.62 | 34.82 |  |
|            |     | $f_F$ | 13.05    | 15.27        | 13.00        | 11.59 | 15.59 | 18.39 | 17.21 |  |
| Loewy      | TE3 | $V_F$ | 45.14    | <b>26.40</b> | <b>11.00</b> | 29.48 | 25.88 | 34.70 | 36.31 |  |
|            |     | $f_F$ | 14.95    | <b>0.283</b> | <b>0.151</b> | 11.57 | 14.36 | 17.64 | 17.16 |  |
|            | TE4 | $V_F$ | 39.69    | <b>25.03</b> | <b>10.80</b> | 26.57 | 25.03 | 33.08 | 34.33 |  |
|            |     | $f_F$ | 13.00    | <b>0.270</b> | <b>0.150</b> | 9.871 | 13.94 | 16.80 | 16.38 |  |



**Figure 5.45** Frequency vs rotational speed for various lamination schemes. TE3.



**Figure 5.46** MAC values at different rotational speeds for the graph 5.45-b.



**Figure 5.47** Real and imaginary parts of Loewy's deficiency function vs reduced frequency for  $\theta = -15$ . In the box, the values for the first reduced frequency.

## 5.6 Rotating plates in a supersonic flow

In this section, the dynamics of a rotating cantilevered plate in a supersonic flow is studied. A number of analysis is carried out considering beams with different values of length-to-thickness ( $L/h$ ) and length-to-chord ( $L/c$ ) ratios. The results are obtained for isotropic structures with Young's modulus  $E=70$  GPa, Poisson's ratio  $\nu=0.3$  and density  $\rho=2700$  kg/m<sup>3</sup>. The air density  $\rho_a$  and the speed of sound  $V_s$  are assumed to be 24.08 kg/m<sup>3</sup> and 344 m/s and, in order to fulfill the condition of supersonic flow on the whole blade surface, the rotational minimum speed ( $\Omega_{min}$ ) must be  $1.8 \frac{V_{sound}}{r_h}$ . To compare the results presented against results in (Farhadi and Hosseini-Hashemi, 2011), where the piston theory is combined with a non-linear Mindlin first-order shear deformation plate theory. For a thickness-to-length ratio equal to 0.015 and 0.01, Tables 5.39 and 5.40 shows the frequency parameters ( $\bar{\omega} = \omega L^2 \sqrt{\frac{12\rho(1-\nu^2)}{Ech}}$ ) for length-to-width ratio equal to 3 and 2.5, respectively.

**Table 5.39** Non-dimensional out-of-plane frequencies for rotating isotropic plate ( $h/L = 0.015$ ,  $r_h/L = 1$ ) at  $\Omega_{min}$ .

| L/c                         | Theory    | $\bar{\omega}_1$ | $\bar{\omega}_2$ | $\bar{\omega}_3$ | $\bar{\omega}_4$ | $\bar{\omega}_5$ | $\bar{\omega}_6$ | $\bar{\omega}_7$ | $\bar{\omega}_8$ |
|-----------------------------|-----------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|
| Free rotation               |           |                  |                  |                  |                  |                  |                  |                  |                  |
| 3                           | TE3       | 42.9             | 48.1             | 103.4            | 125.5            | 175.5            | 211.0            | 243.7            | 266.0            |
|                             | TE4       | 42.9             | 48.0             | 103.4            | 125.2            | 175.3            | 210.1            | 210.3            | 261.7            |
|                             | TE5       | 42.9             | 48.0             | 103.4            | 125.0            | 175.3            | 209.3            | 209.7            | 258.9            |
|                             | 2D (2011) | 41.7             | 52.8             | 100.0            | 125.0            | 172.2            | 208.3            | 211.1            | 255.6            |
| 2.5                         | TE3       | 42.9             | 46.7             | 103.4            | 119.9            | 170.0            | 180.8            | 201.9            | 229.0            |
|                             | TE4       | 42.9             | 46.6             | 103.4            | 119.6            | 150.5            | 176.6            | 201.0            | 214.3            |
|                             | TE5       | 42.9             | 46.6             | 103.4            | 119.4            | 149.4            | 176.6            | 200.6            | 208.1            |
|                             | 2D (2011) | 41.7             | 55.6             | 100.0            | 119.4            | 150.0            | 172.2            | 200.0            | 211.1            |
| Rotation in air medium flow |           |                  |                  |                  |                  |                  |                  |                  |                  |
| 3                           | TE3       | 43.8             | 46.9             | 103.2            | 125.4            | 175.4            | 211.1            | 243.7            | 265.9            |
|                             | TE4       | 43.8             | 46.8             | 103.2            | 125.2            | 175.1            | 209.8            | 210.8            | 261.6            |
|                             | TE5       | 43.8             | 46.8             | 103.2            | 125.0            | 175.1            | 209.0            | 210.2            | 258.8            |
|                             | 2D (2011) | 41.7             | 52.8             | 100.0            | 125.0            | 172.2            | 208.3            | 216.7            | 256.1            |
| 2.5                         | TE3       | 44.5             | 44.8             | 103.2            | 119.9            | 169.9            | 180.7            | 202.0            | 229.0            |
|                             | TE4       | 44.5             | 44.8             | 103.2            | 119.6            | 150.6            | 176.4            | 201.3            | 214.3            |
|                             | TE5       | 44.5             | 44.8             | 103.2            | 119.5            | 149.7            | 176.3            | 200.8            | 208.1            |
|                             | 2D (2011) | 41.7             | 52.8             | 100.0            | 119.4            | 152.8            | 172.2            | 200.0            | 208.3            |

Discrepancies are recorded in the computation of the second frequency ' $\bar{\omega}_2$ ' that is, for all studied configurations, related to the torsional mode. Moreover, for  $h/L$  equal to 0.01, the 1D theories predict frequency lockings between the first and second values for both  $L/c$  ratios and between the third and fourth for the lowest  $L/c$  value. These discrepancies are due to the fact that the reference solution was obtained by a plate theory, which takes into account non-linear strain terms that can play an important role especially for highly deformable plates. However, the comparisons confirms that the 1D-CUF elements provide quite accurate descriptions of the aeroelastic behavior of thin structures rotating in a flow.

Similarly to the subsonic case, the flutter instability occurs when a mode is self-excited that is,

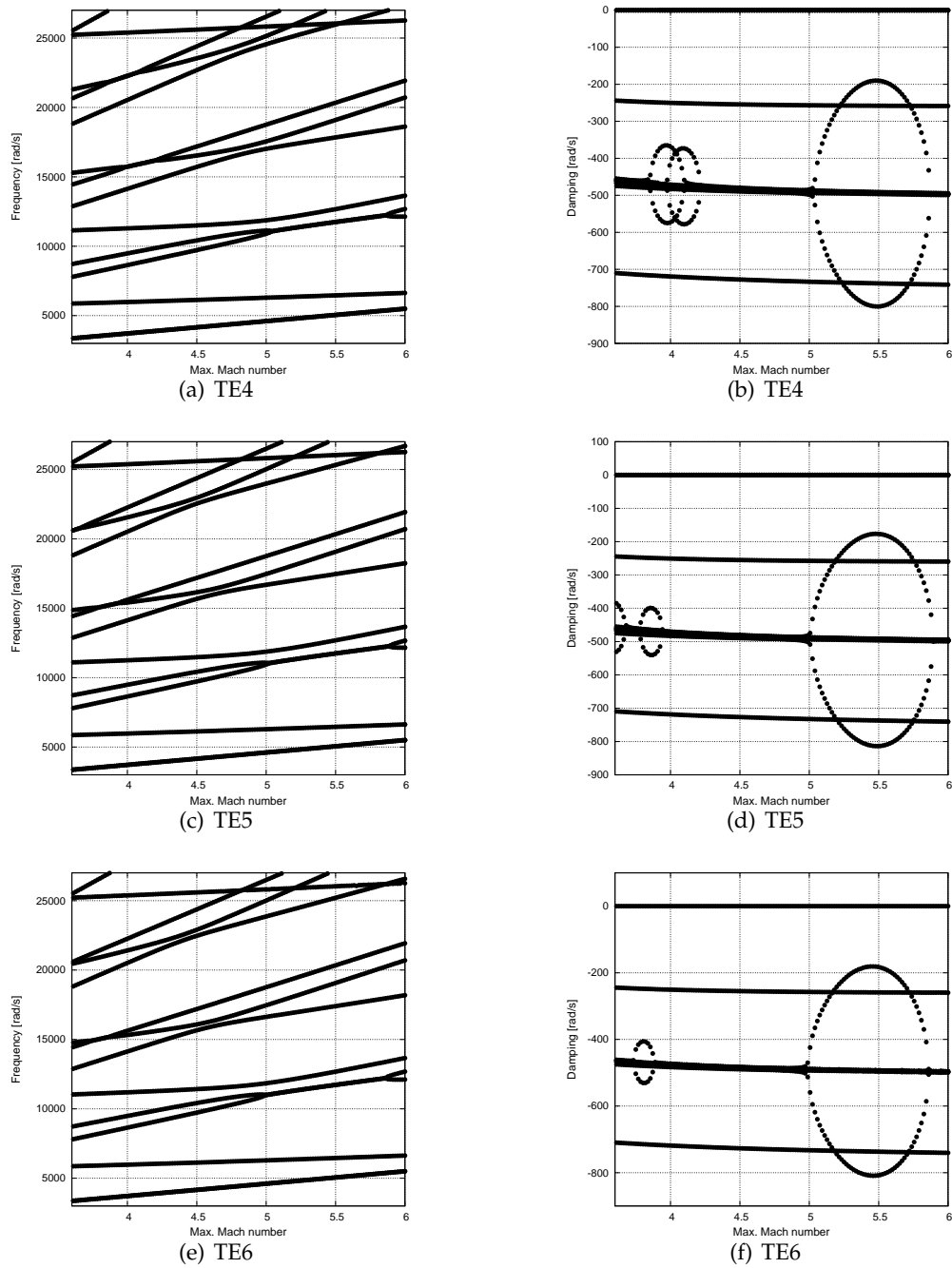
**Table 5.40** Non-dimensional out-of-plane frequencies for rotating isotropic plate ( $h/L = 0.01$ ,  $r_h/L = 1$ ) at  $\Omega_{min}$ .

| L/c                         | Theory    | $\bar{\omega}_1$ | $\bar{\omega}_2$ | $\bar{\omega}_3$ | $\bar{\omega}_4$ | $\bar{\omega}_5$ | $\bar{\omega}_6$ | $\bar{\omega}_7$ | $\bar{\omega}_8$ |
|-----------------------------|-----------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|
| Free rotation               |           |                  |                  |                  |                  |                  |                  |                  |                  |
| 3                           | TE3       | 63.9             | 67.7             | 152.4            | 169.8            | 244.8            | 257.6            | 280.7            | 319.2            |
|                             | TE4       | 63.9             | 67.6             | 152.4            | 169.5            | 216.2            | 251.8            | 279.8            | 297.9            |
|                             | TE5       | 63.9             | 67.6             | 152.4            | 169.4            | 214.7            | 251.8            | 279.4            | 289.5            |
|                             | 2D (2011) | 63.3             | 75.9             | 151.9            | 170.9            | 221.5            | 246.8            | 275.3            | 288.0            |
| 2.5                         | TE3       | 63.9             | 66.6             | 152.2            | 165.2            | 181.8            | 250.3            | 260.5            | 272.8            |
|                             | TE4       | 63.9             | 66.6             | 151.8            | 160.3            | 164.9            | 246.0            | 253.6            | 271.8            |
|                             | TE5       | 63.9             | 66.6             | 151.6            | 158.7            | 164.8            | 238.9            | 252.3            | 271.5            |
|                             | 2D (2011) | 63.3             | 76.6             | 151.3            | 164.6            | 170.9            | 240.5            | 250.6            | 269.0            |
| Rotation in air medium flow |           |                  |                  |                  |                  |                  |                  |                  |                  |
| 3                           | TE3       | 65.3             | 65.5             | 151.7            | 169.7            | 244.3            | 256.9            | 281.6            | 319.0            |
|                             | TE4       | 65.3             | 65.6             | 151.7            | 169.7            | 217.0            | 250.6            | 281.2            | 297.7            |
|                             | TE5       | 65.4             | 65.7             | 151.8            | 169.8            | 216.2            | 250.6            | 280.9            | 289.5            |
|                             | 2D (2011) | 63.3             | 69.6             | 145.6            | 170.9            | 224.7            | 237.3            | 278.5            | 291.1            |
| 2.5                         | TE3       | 64.8             | 65.0             | 152.1            | 163.6            | 182.7            | 248.6            | 261.2            | 273.3            |
|                             | TE4       | 65.0             | 65.2             | 154.7            | 154.8            | 168.9            | 244.7            | 253.5            | 273.3            |
|                             | TE5       | 65.1             | 65.3             | 154.5            | 154.6            | 169.2            | 238.7            | 251.4            | 273.3            |
|                             | 2D (2011) | 63.3             | 75.9             | 145.6            | 167.7            | 183.5            | 237.3            | 246.8            | 272.2            |

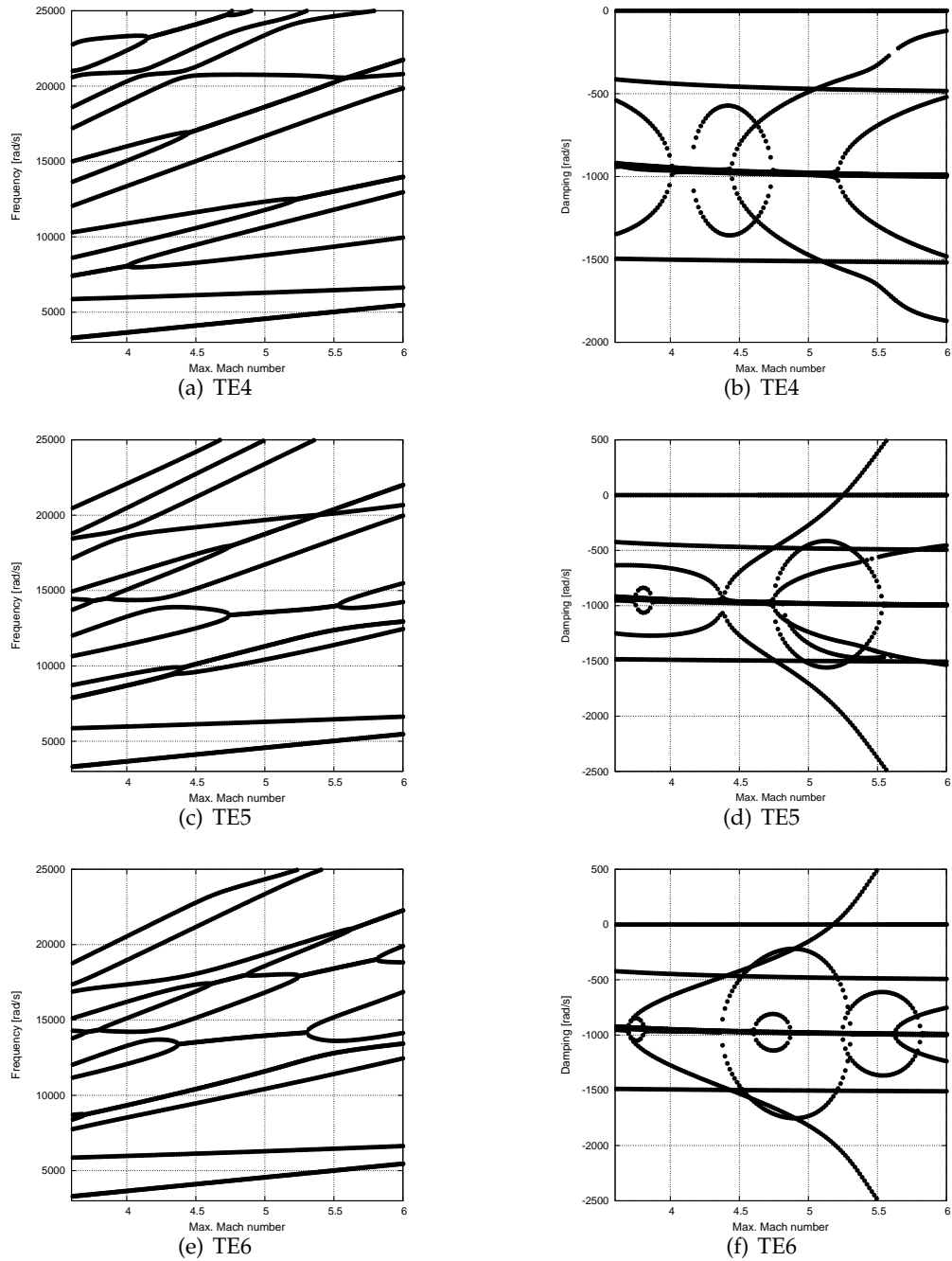
the real part of the related eigenvalue becomes positive. The instability condition depends on several parameters such as structure dimensions and material properties. In Figs.5.48 and 5.49, frequency and damping values of isotropic plates, with length and chord equal to 0.3 and 0.1 m, are shown as functions of the Mach number at the structure tip ( $2 \times L$ ) for ' $h/L$ ' equal to 0.01 and 0.005, respectively. For both cases, TE4, TE5 and TE6 displacement theories are used. Looking at the frequency plots, veering, crossing as well as locking phenomena are clearly recognizable for both structures. When the blocking phenomenon occurs, the real parts of the involved eigenvalues diverge and, if one of them becomes positive, the system is unstable. Results reveal that the thinner plate becomes unstable when the maximum velocity reaches a Mach number equal to 5.3, while in contrast, the thicker plate remains stable for all considered speeds. It should be noted that the TE4 expansion does not predict any instability region for the thinner plate.

In the conclusive analyses, we consider the same thinner plate of the previous example (' $h/L=0.005$ ') constituted by two layers of orthotropic materials. The material properties are  $E_{11}=98$  GPa,  $E_{22}=E_{33}=7.90$  GPa,  $\nu_{12}=\nu_{31}=\nu_{23}=0.28$ ,  $G_{12}=G_{31}=G_{23}=5.60$  GPa, and  $\rho=1520$  kg/m<sup>3</sup>. The lamination sequence is anti-symmetric with respect to the mean plane of the plate [ $\theta/-\theta$ ]. Figure 5.50 shows the evolutions of frequency and damping values with the rotational speed. The analyses are carried out using the Taylor-like expansion of sixth-order.

It should be noted that damping values relating to the first 15 mode shapes are negative in the whole speed range considered only for  $\theta=75^\circ$  and  $90^\circ$ . This fact demonstrates that a correct choice of the lamination sequence can be an effective way to avoid flutter instabilities, especially when high deformable structures must be used.

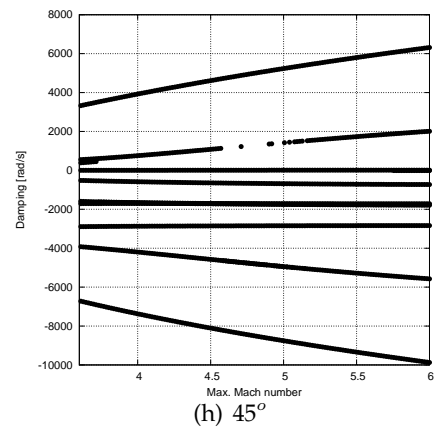
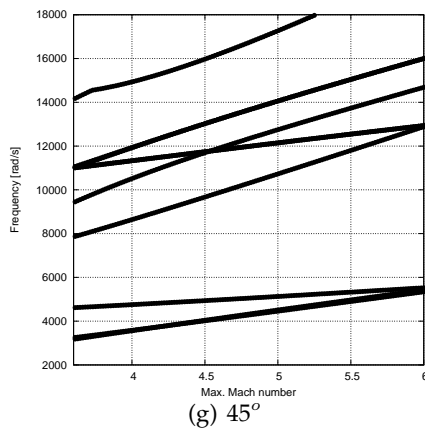
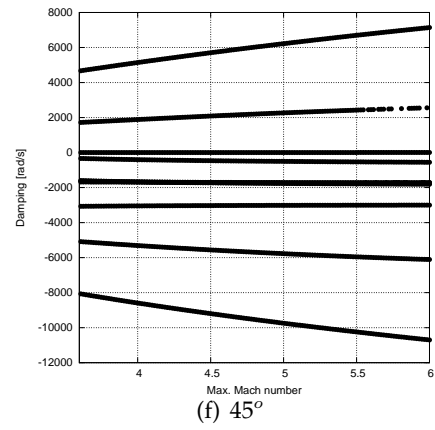
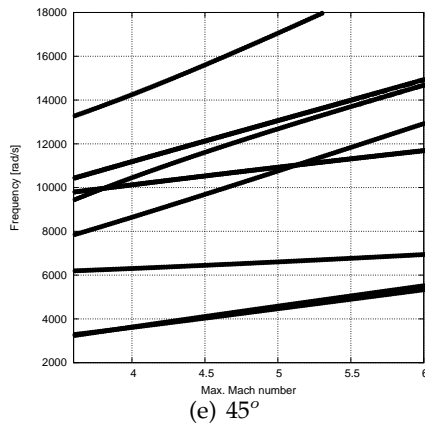
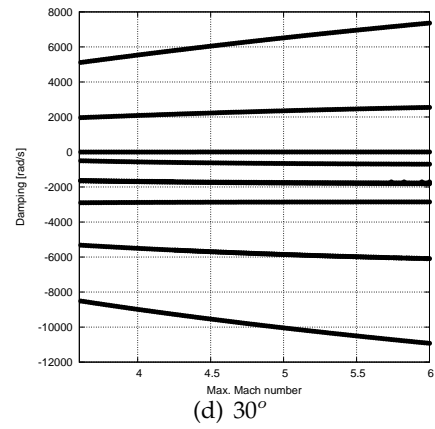
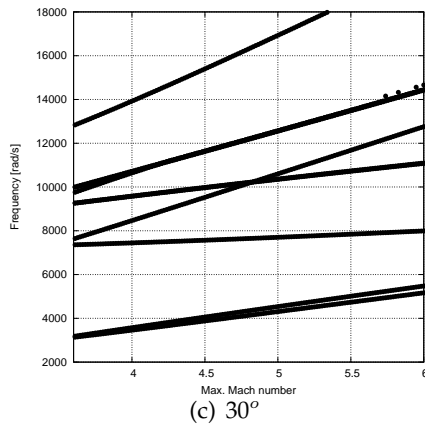
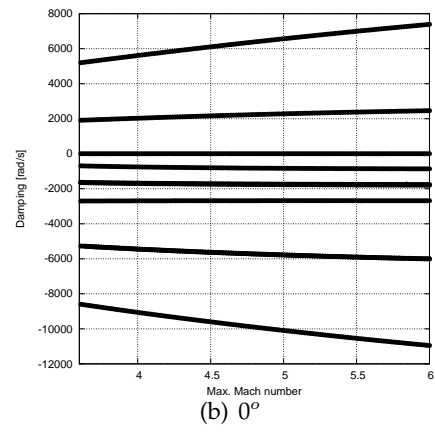
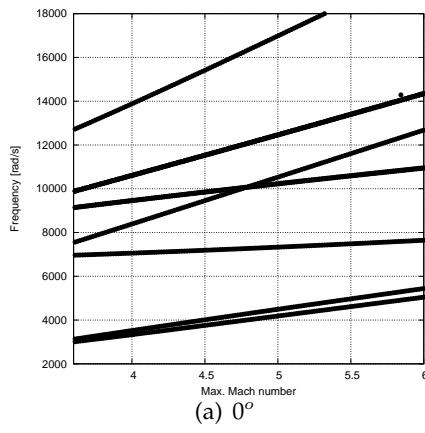


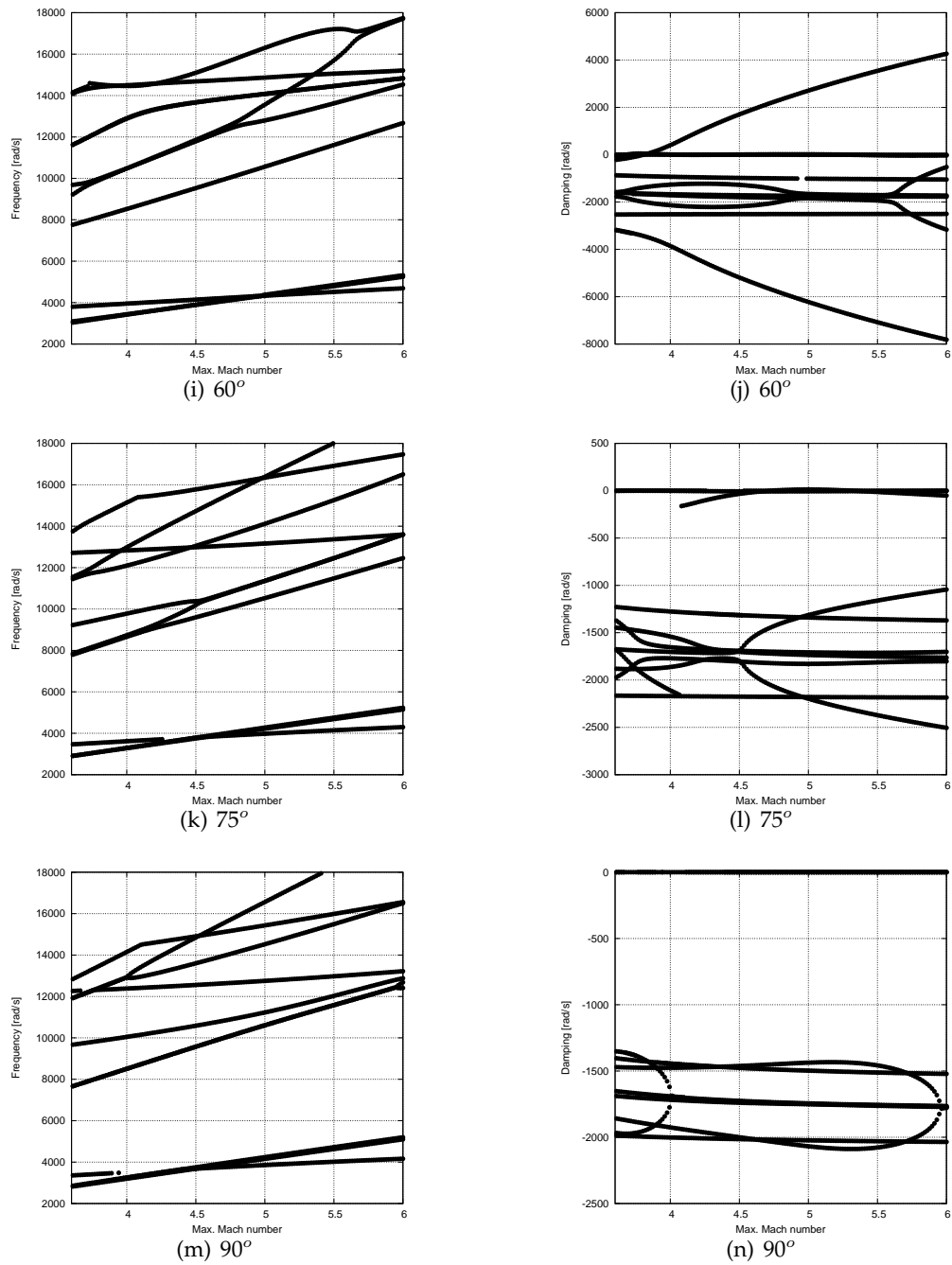
**Figure 5.48** Frequency and damping values of an isotropic plate rotating in compressed air with  $h/L$  equal to 0.01.



**Figure 5.49** Frequency and damping values of an isotropic plate rotating in compressed air with  $h/L$  equal to 0.005.







**Figure 5.50** Frequency and damping values of a laminated plate rotating in compressed air with  $h/L$  equal to 0.005. (TE6)



## Chapter 6

# Conclusions

This thesis aims to present a simple tool for the aeroelastic analysis of rotating structures. Both steady and unsteady aerodynamic theories are coupled with refined beam theories obtained through Carrera Unified Formulation. The enhanced capabilities of 1D CUF theories allowed to deal with problems that usually require sophisticated solutions with a low number of degrees of freedom.

### 6.1 Outline and contribution to the literature

The FE and CUF approximations allow the development of several refined beam finite elements. Besides Taylor- and Lagrange-like expansions (TE and LE), the displacement fields are obtained by using several functions including trigonometric, exponential, hyperbolic and zig-zag terms. In order to verify the effectiveness of the finite elements, linear static and free-vibration analyses have been carried out on structures made of composite as well as functionally graded material. In the light of the obtained results, it is clear that

- although the classical beam theories provide acceptable results in terms of displacements and longitudinal stress, they do not describe the correct shear stress distribution, especially when shear effects become relevant;
- the use of refined 1D models is recommended in order to achieve acceptable results when thick structures are considered;
- the zig-zag function significantly improves the solution when thick laminated and sandwich beams are studied;
- 1D CUF beam elements provide results very close to shell and solid finite element solutions;
- the accuracy of the solution strictly depends on the problem characteristics, therefore the possibility to develop an (at least) infinite number of theories appears to be a remarkable feature of the present formulation.

In the second part of this work we have focused on the study of rotors using CUF. The equations of motion, derived through Hamilton's Principle, include both Coriolis term and centrifugal

contributions. The TE elements have been used to perform a number of free-vibration analyses on spinning shafts and blades made of either isotropic or composite materials. Moreover, the higher-order models have allowed evaluation of the dynamic behavior of highly deformable spinning structures such as thin-walled cylinders. To sum up:

- regarding spinning shafts, classical beam theories provide accurate results when slender and metallic structures are considered;
- since the bending-torsional coupling must be described in order to correctly predict the occurrence of instabilities, the higher-order theories are recommended for the study of shafts made of composite materials;
- the initial stress is essential when the wall thickness becomes thin, confirming that classical beam theories are suitable only to study compact cross-sections;
- refined beam models provide reliable results for thick laminated cylindrical shells;
- as for the centrifugally stiffened beams, the present formulation is able to describe both chordwise and flapwise motions;
- 1D CUF elements leads to more accurate results than classical theories when laminated and thin-walled blades are studied;
- since stretching and chordwise motions are coupled by Coriolis matrix, the non-linear term ' $u_{y,y}$ ' introduces a stiffness contribution that can be important for the prediction of divergences.

In the third section, aerodynamic theories have been developed and combined with the refined beam theories. Theodorsen's approach (from which the steady and quasi-steady theories can be easily obtained) has been used to evaluate flutter conditions of various wing configurations including straight and swept structures constituted by isotropic and composite materials. Due to their wide use in the modelling of wings and blades, special attention has been given to thin-walled laminated beams. Flutter speeds and frequencies have been compared with those obtained with the DLM and, with solutions available in the literature, yielding the following findings:

- since the flutter phenomenon involves a bending-torsion coupling, refined beam models are needed to predict an accurate dynamic response of the structure;
- while the steady approach predicts too low flutter velocity values, the unsteady strip theory generally yields results very close to the reference values;
- as expected, the lamination scheme and the sweep angle affect the dynamics of the structures to a great extent. As a consequence, a correct setting of these two parameters is crucial for the aeroelastic design.

The natural extension of Theodorsen's model for the rotary-wing aeroelasticity, namely Loewy's theory, has been then embedded in CUF framework writing both structural and aerodynamic matrices in terms of *fundamental nuclei*. The differences between classical aerodynamic theories for fixed wings and Loewy's approach have been investigated by analysing metallic and nonhomogeneous blades:

- as expected, the steady approach leads to more conservative results than unsteady models in terms of flutter velocities;
- although different values of the spacing factor ' $\hat{h}$ ' have been considered, Loewy's theory provides results very close to those obtained with Theodorsen's approach for the metallic blade;
- as expected, the stacking sequence strongly affects the aeroelastic response of blades;
- since the inclusion of the returning wave implies the possibility of single-degree-of-freedom flutter in blade torsion, considerable discrepancies between Loewy's and Theodorsen's theory are observed when the nonhomogeneous blade is considered;
- the spacing factor ' $\hat{h}$ ' can affect the solution, therefore it is a crucial parameter for instability predictions.

Finally, the Piston Theory combined with the higher-order structural theories has been adopted in aeroelastic analyses of plates rotating in compressed and supersonic flows. The effects of aspect-ratios and lamination scheme have been evaluated and the results were reported in terms of natural frequencies. In the light of these results, the following conclusions are drawn:

- refined CUF models strongly agree with the reference solutions (obtained with a non-linear plate theory) for the moderately thick rotating plate;
- for the thinnest plate, slight discrepancies are observed in the computation of torsional frequencies and, therefore, frequency blocking phenomena predicted at different rotational speed;
- higher order theories are needed to correctly predict the instability conditions especially when thin structures are considered;
- an adequate choice of the stacking sequence can avoid the occurrence of instabilities.

## 6.2 Future works

Within this work, Carrera Unified Formulation has shown excellent performance in the study of the dynamics and the aeroelasticity of rotors, therefore paving the way for future research that will include:

1. extension to nonlinear structural analysis (geometrical as well as material nonlinearities);
2. extension to multi-field problems in order to evaluate, for instance, the temperature effects on turbine blades;
3. development of cascade theory for turbine and compressor blades in subsonic and supersonic flows;
4. analysis of complex-shaped rotating structures (swept-tip blades, twisted blades etc.);
5. extension to more sophisticated aerodynamic theories (panel methods, CFD)

## Appendix A

In the following section, we present the coefficients of the transformed material matrix  $\tilde{C}$  as a function of stiffness material coefficients  $C$  and the two angles  $\theta, \beta$ .

### Isotropic and orthotropic materials

$$\begin{aligned}
\tilde{C}_{33} &= C_{33}\cos^4\theta + 2(C_{23} + 2C_{66})\cos^2\theta\sin^2\theta + C_{22}\sin^4\theta \\
\tilde{C}_{32} &= \cos^2\beta(C_{23}(\cos^4\theta + \sin^4\theta) + (C_{33} + C_{22} - 4C_{66})\cos^2\theta\sin^2\theta) + \sin^2\beta(C_{13}\cos^2\theta + C_{21}\sin^2\theta) \\
\tilde{C}_{31} &= \sin^2\beta(C_{23}(\cos^4\theta + \sin^4\theta) + (C_{33} + C_{22} - 4C_{66})\cos^2\theta\sin^2\theta) + \cos^2\beta(C_{13}\cos^2\theta + C_{21}\sin^2\theta) \\
\tilde{C}_{36} &= \cos\beta((2C_{66} + C_{23} - C_{33})\cos^3\theta\sin\theta + (-2C_{66} + C_{22} - C_{23})\sin^3\theta\cos\theta) \\
\tilde{C}_{22} &= \cos^4\beta(C_{22}\cos^4\theta + (4C_{66} + 2C_{23})\cos^2\theta\sin^2\theta + C_{33}\sin^4\theta) + \\
&\quad + \cos^2\beta\sin^2\beta((4C_{44} + 2C_{21})\cos^2\theta + (4C_{55} + 2C_{13})\sin^2\theta) + C_{11}\sin^4\beta \\
\tilde{C}_{12} &= \cos^2\beta\sin^2\beta((4C_{66} + 2C_{23})\cos^2\theta\sin^2\theta + C_{33}\sin^4\theta - 4C_{55}\sin^2\theta + C_{22}\cos^4\theta - 4C_{44}\cos^2\theta + C_{11}) + \\
&\quad + (\cos^4\beta + \sin^4\beta)(C_{13}\sin^2\theta + C_{21}\cos^2\theta) \\
\tilde{C}_{26} &= \cos^3\beta((2C_{66} + C_{23} - C_{33})\sin^3\theta\cos\theta + (-2C_{66} + C_{22} - C_{33})\cos^3\theta\sin\theta) + \\
&\quad + \cos\beta\sin^2\beta((2C_{44} - 2C_{55} + C_{21} - C_{13})\cos\theta\sin\theta) \\
\tilde{C}_{11} &= C_{11}\cos^4\beta + \cos^2\beta\sin^2\beta((4C_{44} + 2C_{21})\cos^2\theta + (4C_{55} + 2C_{13})\sin^2\theta) + \\
&\quad + \sin^4\beta(C_{22}\cos^4\theta + C_{33}\sin^4\theta + (4C_{66} + 2C_{23})\cos^2\theta\sin^2\theta) \\
\tilde{C}_{16} &= \cos^3\beta(C_{21} - C_{13})\cos\theta\sin\theta + \cos\beta\sin^2\beta((2C_{66} + C_{23} - C_{33})\cos\theta\sin^3\theta + \\
&\quad + (-2C_{66} + C_{22} - C_{23})\cos^3\theta\sin\theta) \\
\tilde{C}_{44} &= (\cos^4\beta + \sin^4\beta)(C_{44}\cos^2\theta + C_{55}\sin^2\theta) + \cos^2\beta\sin^2\beta(C_{11} + C_{22}\cos^4\theta + \\
&\quad + (-2C_{44} - 2C_{12})\cos^2\theta + (-2C_{55} - 2C_{13})\sin^2\theta + (4C_{66} + 2C_{23})\cos^2\theta\sin^2\theta) \\
\tilde{C}_{45} &= \cos^3\beta(C_{44} - C_{55})\cos\theta\sin\theta + \cos\beta\sin^2\beta((-2C_{66} + C_{22} - C_{23})\cos^3\theta\sin\theta + \\
&\quad + (2C_{66} + C_{23} - C_{33})\cos\theta\sin^3\theta + (C_{55} - C_{44} - C_{12} + C_{13})) \\
\tilde{C}_{55} &= \cos^2\beta(C_{55}\cos^2\theta + C_{44}\sin^2\theta) + \sin^2\beta(C_{66}(\cos^4\theta + \sin^4\theta) + \\
&\quad + (C_{33} + C_{22} - 2C_{66} - 2C_{23})\cos^2\theta\sin^2\theta) \\
\tilde{C}_{66} &= \sin^2\beta(C_{55}\cos^2\theta + C_{44}\sin^2\theta) + \cos^2\beta(C_{66}(\cos^4\theta + \sin^4\theta) + \\
&\quad + (C_{33} + C_{22} - 2C_{66} - 2C_{23})\cos^2\theta\sin^2\theta) \\
\tilde{C}_{34} &= \cos\beta\sin\beta((C_{33} + C_{22} - 4C_{66})\cos^2\theta\sin^2\theta + C_{23}(\cos^4\beta + \sin^4\beta) - C_{13}\cos^2\theta - C_{12}\sin^2\theta) \\
\tilde{C}_{35} &= \sin\beta((C_{22} - C_{23} - 2C_{66})\cos\theta\sin^3\theta + (-C_{33} + C_{23} + 2C_{66})\cos^3\theta\sin\theta) \\
\tilde{C}_{24} &= \cos^3\beta\sin\beta(C_{22}\cos^4\theta - (C_{21} + 2C_{44})\cos^2\theta - (4C_{66} + 2C_{23})\cos^2\theta\sin^2\theta + \\
&\quad + (C_{13} + 2C_{55})\sin^2\theta + C_{33}\sin^4\theta) + \cos\beta\sin^3\beta((C_{13} + 2C_{55})\sin^2\theta + (C_{12} + 2C_{44})\cos^2\theta - C_{11}) \\
\tilde{C}_{25} &= \sin^3\beta(C_{21} - C_{13})\cos\theta\sin\theta + \cos\beta\sin^3\beta((2C_{66} + C_{23} - C_{33})\cos\theta\sin^3\theta + \\
&\quad + (-2C_{66} - C_{23} + C_{22})\cos^3\theta\sin\theta + (2C_{55} - 2C_{44})\cos\theta\sin\theta) \\
\tilde{C}_{14} &= \cos\beta\sin^3\beta(C_{22}\cos^4\theta - (C_{21} + 2C_{44})\cos^2\theta - (4C_{66} + 2C_{23})\cos^2\theta\sin^2\theta + \\
&\quad + (C_{13} + 2C_{55})\sin^2\theta + C_{33}\sin^4\theta) + \cos^3\beta\sin\beta((C_{13} + 2C_{55})\sin^2\theta + (C_{12} + 2C_{44})\cos^2\theta - C_{11}) \\
\tilde{C}_{15} &= \sin^3\beta((2C_{66} + C_{23} - C_{33})\cos\theta\sin^3\theta + (-2C_{66} - C_{23} + C_{22})\cos^3\theta\sin\theta) + \\
&\quad + \cos^2\beta\sin\beta(2C_{44} - 2C_{55} + C_{21} - C_{13})\cos\theta\sin\theta \\
\tilde{C}_{45} &= \sin^3\beta(C_{44} - C_{55})\cos\theta\sin\theta + \cos^2\beta\sin\beta((2C_{66} + C_{23} - C_{33})\cos\theta\sin^3\theta + \\
&\quad + (-2C_{66} - C_{23} + C_{22})\cos^3\theta\sin\theta + (C_{55} - C_{44} - C_{21} + C_{13})\cos\theta\sin\theta) \\
\tilde{C}_{56} &= \cos\beta\sin\beta(C_{66}(\cos^4\theta + \sin^4\theta) + (C_{22} + C_{33} - 2C_{23} - 2C_{66})\cos^2\theta\sin^2\theta - C_{44}\sin^2\theta - C_{55}\cos^2\theta)
\end{aligned} \tag{1}$$

### Functionally Graded Materials

$$\tilde{C}_{11}^k = \tilde{C}_{22}^k = \tilde{C}_{33}^k = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \quad \tilde{C}_{44}^k = \tilde{C}_{55}^k = \tilde{C}_{66}^k = \frac{E}{2(1+\nu)} \tag{2}$$

$$\tilde{C}_{12}^k = \tilde{C}_{13}^k = \tilde{C}_{23}^k = \frac{E\nu}{(1+\nu)(1-2\nu)} \tag{3}$$

## Appendix B





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# Curriculum Vitae

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## Education

- 2012 - Present - **PhD in Fluid Dynamics**, Politecnico di Torino, Turin, Italy. PhD Topic: Aeroelasticity.
- 2009 - 2011 - **MSc in Aerospace Engineering**, Politecnico di Torino, Turin, Italy. Master variant: Aerospace Structures. Thesis title: *Advanced one-dimensional models for the analyses of reinforced shell structures*.
- 2003 - 2009 - **BSc in Aerospace Engineering**, Politecnico di Torino, Turin, Italy. Thesis title: *Structural study of a propeller*.

## Publications

### International Journals and Book Chapters

- E. Carrera **M. Filippi** (2015) "Vibration analysis of thin/thick, composites/metallic spinning cylindrical shells by refined beam models", Journal of Vibration and Acoustics - ASME DC, DOI: <http://dx.doi.org/10.1115/1.4029688>, Accepted Manuscript.
- **M. Filippi**, E. Carrera, A.M. Zenkour (2014) "Static Analyses of FGM Beams by Various Theories and Finite Elements", Composites: Part B, Vol. 72, pp. 1–9. DOI: <http://dx.doi.org/10.1016/j.compositesb.2014.12.004>.
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- D.S. Mashat, E. Carrera, A.M. Zenkour, S.A. Al Khateeb, **M. Filippi** (2014) “Free Vibration of FGM Layered Beams by Various Theories and Finite Elements”, *Composites: Part B*, Vol. 59, pp. 269–278.  
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